CSE 114A

Introduction to Functional Programming

Higher-Order Functions

Based on course materials developed by Nadia Polikarpova and Owen Arden

Plan for this week

Last week:

- user-defined *data types*
  - and how to manipulate them using *pattern matching* and *recursion*
- how to make recursive functions more efficient with *tail recursion*

This week:

- code reuse with *higher-order functions* (HOFs)
- some useful HOFs: `map`, `filter`, and `fold`

Recursion is good

- Recursive code mirrors recursive data
  - Base constructor -> Base case
  - Inductive constructor -> Inductive case (with recursive call)
- But it can get kinda repetitive!

Example: evens

Let's write a function evens:

```
-- evens []          ==> []
-- evens [1,2,3,4] ==> [2,4]
evens        :: [Int] -> [Int]
evens []      = ...
evens (x:xs) = ...
```

Example: four-letter words

Let's write a function fourChars:

```
-- fourChars [] ==> []
-- fourChars ["i","must","do","work"] ==> ["must","work"]
fourChars :: [String] -> [String]
fourChars []     = ...
fourChars (x:xs) = ...
```

Yikes, Most Code is the Same!

```
foo []              = []
foo (x:xs)
   | x mod 2 == 0   = x : foo xs
   | otherwise      =     foo xs

foo []              = []
foo (x:xs)
   | length x == 4  = x : foo xs
   | otherwise      =     foo xs
```

Only difference is **condition**

- `x mod 2 == 0` vs `length x == 4`

Moral of the day

D.R.Y. Don't Repeat Yourself!

Can we

- *reuse* the general pattern and
- *substitute in* the custom condition?

HOFs to the rescue!

General **Pattern**

- expressed as a *higher-order function*
- takes customizable operations as *arguments*

Specific **Operation**

- passed in as an argument to the HOF

The "filter" pattern

```
evens []            = []
evens (x:xs)
   | x `mod` 2 == 0 = x : evens xs
   | otherwise      =     evens xs
```

```
fourChars []        = []
fourChars (x:xs)
   | length x == 4  = x : fourChars xs
   | otherwise      =     fourChars xs
```

```
filter f []         = []
filter f (x:xs)
   | f x            = x : filter f xs
   | otherwise      =     filter f xs
```

Use the `filter` pattern to avoid duplicating code!

The “filter” pattern

General **Pattern**

- HOF `filter`
- Recursively traverse list and pick out elements that satisfy a predicate

Specific **Operation**

- Predicates `isEven` and `isFour`

```
filter f []          = []
filter f (x:xs)
  | f x              = x : filter f xs
  | otherwise        =     filter f xs
```

```
evens          = filter isEven
  where
    isEven x = x `mod` 2 == 0
```

```
fourChars      = filter isFour
  where
    isFour x = length x == 4
```

Let's talk about types

```haskell
-- evens [1,2,3,4] ==> [2,4]
evens :: [Int] -> [Int]
evens xs = filter isEven xs
  where
    isEven :: Int -> Bool
    isEven x  =  x `mod` 2 == 0
filter :: ???
```

Let’s talk about types

```
-- evens [1,2,3,4] ==> [2,4]
evens :: [Int] -> [Int]
evens xs = filter isEven xs
  where
    isEven :: Int -> Bool
    isEven x  =  x `mod` 2 == 0
filter :: ???
```

Let’s talk about types

```
-- fourChars ["i","must","do","work"] ==> ["must","work"]
fourChars :: [String] -> [String]
fourChars xs = filter isFour xs
  where
    isFour :: String -> Bool
    isFour x  =  length x == 4
filter :: ???
```

Let's talk about types

Uh oh! So what's the type of `filter`?

```
filter :: (Int -> Bool) -> [Int] -> [Int] -- ???
```

```
filter :: (String -> Bool) -> [String] -> [String] -- ???
```

- It *does not care* what the list elements are
  - as long as the predicate can handle them
- It's type is **polymorphic** (generic) in the type of list elements

```
-- For any type `a`
--   if you give me a predicate on `a`s
--   and a list of `a`s,
--   I'll give you back a list of `a`s
filter :: (a -> Bool) -> [a] -> [a]
```

Example: all caps

Lets write a function `shout`:

```
-- shout []                    ==> []
-- shout ['h','e','l','l','o'] ==> ['H','E','L','L','O']
shout :: [Char] -> [Char]
shout []     = ...
shout (x:xs) = ...
```

Example: squares

Lets write a function `squares`:

```
-- squares []        ==> []
-- squares [1,2,3,4] ==> [1,4,9,16]
squares :: [Int] -> [Int]
squares []     = ...
squares (x:xs) = ...
```

Yikes, Most Code is the Same!

Lets rename the functions to `foo`:

```
-- shout
foo []     = []
foo (x:xs) = toUpper x : foo xs

-- squares
foo []     = []
foo (x:xs) = (x * x)   : foo xs
```

Lets **refactor** into the **common pattern**

```
pattern = ...
```

The “map” pattern

```
shout []     = []
shout (x:xs) = toUpper x : shout xs
```

```
squares []     = []
squares (x:xs) = (x*x) : squares xs
```

```
map f []     = []
map f (x:xs) = f x : map f xs
```

The `map` Pattern

General Pattern

- HOF `map`
- Apply a transformation `f` to each element of a list

Specific Operations

- Transformations `toUpper` and `\x -> x * x`

The "map" pattern

```
map f []     = []
map f (x:xs) = f x : map f xs
```

Lets refactor `shout` and `squares`

```
shout   = map ...

squares = map ...
```

```
map f []     = []
map f (x:xs) = f x : map f xs
```

```
shout = map (\x -> toUpper x)
```

```
squares = map (\x -> x*x)
```

QUIZ

What is the type of map? *

```
map f []     = []
map f (x:xs) = f x : map f xs
```

(A) `(Char -> Char) -> [Char] -> [Char]`

(B) `(Int -> Int) -> [Int] -> [Int]`

(C) `(a -> a) -> [a] -> [a]`

(D) `(a -> b) -> [a] -> [b]`

(E) `(a -> b) -> [c] -> [d]`

http://tiny.cc/cse116-map-ind

QUIZ

What is the type of map? *

```
map f []     = []
map f (x:xs) = f x : map f xs
```

(A) `(Char -> Char) -> [Char] -> [Char]`

(B) `(Int -> Int) -> [Int] -> [Int]`

(C) `(a -> a) -> [a] -> [a]`

(D) `(a -> b) -> [a] -> [b]`

(E) `(a -> b) -> [c] -> [d]`

http://tiny.cc/cse116-map-grp

The “map” pattern

```
-- For any types `a` and `b`
--   if you give me a transformation from `a` to `b`
--   and a list of `a`s,
--   I'll give you back a list of `b`s
map :: (a -> b) -> [a] -> [b]
```

Type says it all!

- The only meaningful thing a function of this type can do is apply its first argument to elements of the list (Hoogle it!)

Things to try at home:

- can you write a function `map' :: (a -> b) -> [a] -> [b]` whose behavior is different from `map`?
- can you write a function `map' :: (a -> b) -> [a] -> [b]` such that `map' f xs` returns a list whose elements are not in `map f xs`?

QUIZ

What is the value of quiz? *

```
map :: (a -> b) -> [a] -> [b]

quiz = map (\(x, y) -> x + y) [1, 2, 3]
```

- (A) [2, 4, 6]
- (B) [3, 5]
- (C) Syntax Error
- (D) Type Error
- (E) None of the above

http://tiny.cc/cse116-quiz-ind

QUIZ

What is the value of quiz? *

```
map :: (a -> b) -> [a] -> [b]

quiz = map (\(x, y) -> x + y) [1, 2, 3]
```

- (A) [2, 4, 6]
- (B) [3, 5]
- (C) Syntax Error
- (D) Type Error
- (E) None of the above

http://tiny.cc/cse116-quiz-grp

Don’t Repeat Yourself

Benefits of **factoring** code with HOFs:

- Reuse iteration pattern
    - think in terms of standard patterns
    - less to write
    - easier to communicate
- Avoid bugs due to repetition

Recall: length of a list

```
-- len []             ==> 0
-- len ["carne","asada"] ==> 2
len :: [a] -> Int
len []     = 0
len (x:xs) = 1 + len xs
```

Recall: summing a list

```haskell
-- sum []      ==> 0
-- sum [1,2,3] ==> 6
sum :: [Int] -> Int
sum []     = 0
sum (x:xs) = x + sum xs
```

Example: string concatenation

Let's write a function `cat`:

```
-- cat [] ==> ""
-- cat ["carne","asada","torta"] ==> "carneasadatorta"
cat :: [String] -> String
cat []      = ...
cat (x:xs) = ...
```

Can you spot the pattern?

```
-- len
foo []     = 0
foo (x:xs) = 1 + foo xs

-- sum
foo []     = 0
foo (x:xs) = x + foo xs

-- cat
foo []     = ""
foo (x:xs) = x ++ foo xs

pattern = ...
```

The “fold-right” pattern

```
len []     = 0
len (x:xs) = 1 + len xs
```

```
sum []     = 0
sum (x:xs) = x + sum xs
```

```
cat []     = “”
cat (x:xs) = x ++ sum xs
```

```
foldr f b []     = b
foldr f b (x:xs) = f x (foldr f b xs)
```

The `foldr` Pattern

General Pattern

- Recurse on tail
- Combine result with the head using some binary operation

The “fold-right” pattern

```
foldr f b []     = b
foldr f b (x:xs) = f x (foldr f b xs)
```

Let’s refactor `sum`, `len` and `cat`:

```
sum = foldr ...  ...

cat = foldr ...  ...

len = foldr ...  ...
```

Factor the recursion out!

The “fold-right” pattern

```
foldr f b []     = b
foldr f b (x:xs) = f x (foldr f b xs)
```

```
len = foldr (\x n -> 1 + n) 0
```

```
sum = foldr (\x n -> x + n) 0
```

```
cat = foldr (\x s -> x ++ n) “”
```

You can write it more clearly as

```
sum = foldr (+) 0
cat = foldr (++) ""
```

The “fold-right” pattern

```
foldr f b []     = b
foldr f b (x:xs) = f x (foldr f b xs)
```

```
len = foldr (\x n -> 1 + n) 0
```

```
sum = foldr (\x n -> x + n) 0
```

```
cat = foldr (\x s -> x ++ n) “”
```

You can write it more clearly as

```
sum = foldr (+) 0
cat = foldr (++) ""
```

QUIZ

What does this evaluate to? *

```
foldr f b []     = b
foldr f b (x:xs) = f x (foldr f b xs)

quiz = foldr (:) [] [1,2,3]
```

- (A) Type error
- (B) [1,2,3]
- (C) [3,2,1]
- (D) [[3],[2],[1]]
- (E) [[1],[2],[3]]

http://tiny.cc/cse116-foldeval-ind

QUIZ

What does this evaluate to? *

```
foldr f b []     = b
foldr f b (x:xs) = f x (foldr f b xs)

quiz = foldr (:) [] [1,2,3]
```

- (A) Type error
- (B) [1,2,3]
- (C) [3,2,1]
- (D) [[3],[2],[1]]
- (E) [[1],[2],[3]]

http://tiny.cc/cse116-foldeval-grp

The “fold-right” pattern

```
foldr f b []     = b
foldr f b (x:xs) = f x (foldr f b xs)

foldr (:) [] [1,2,3]
  ==> (:) 1 (foldr (:) [] [2, 3])
  ==> (:) 1 ((:) 2 (foldr (:) [] [3]))
  ==> (:) 1 ((:) 2 ((:) 3 (foldr (:) [] [])))
  ==> (:) 1 ((:) 2 ((:) 3 []))
  ==  1 : (2 : (3 : []))
  ==  [1,2,3]
```

The “fold-right” pattern

```
foldr f b [x1, x2, x3, x4]
  ==> f x1 (foldr f b [x2, x3, x4])
  ==> f x1 (f x2 (foldr f b [x3, x4]))
  ==> f x1 (f x2 (f x3 (foldr f b [x4])))
  ==> f x1 (f x2 (f x3 (f x4 (foldr f b []))))
  ==> f x1 (f x2 (f x3 (f x4 b)))
```

Accumulate the values from the **right**

For example:

```
foldr (+) 0 [1, 2, 3, 4]
  ==> 1 + (foldr (+) 1 [2, 3, 4])
  ==> 1 + (2 + (foldr (+) 0 [3, 4]))
  ==> 1 + (2 + (3 + (foldr (+) 0 [4])))
  ==> 1 + (2 + (3 + (4 + (foldr (+) 0 []))))
  ==> 1 + (2 + (3 + (4 + 0)))
```

QUIZ

What is the most general type of foldr? *

```
foldr f b []     = b
foldr f b (x:xs) = f x (foldr f b xs)
```

- (A) (a -> a -> a) -> a -> [a] -> a
- (B) (a -> a -> b) -> a -> [a] -> b
- (C) (a -> b -> a) -> b -> [a] -> b
- (D) (a -> b -> b) -> b -> [a] -> b
- (E) (b -> a -> b) -> b -> [a] -> b

http://tiny.cc/cse116-foldtype-ind

QUIZ

What is the most general type of foldr? *

```
foldr f b []     = b
foldr f b (x:xs) = f x (foldr f b xs)
```

- (A) (a -> a -> a) -> a -> [a] -> a
- (B) (a -> a -> b) -> a -> [a] -> b
- (C) (a -> b -> a) -> b -> [a] -> b
- (D) (a -> b -> b) -> b -> [a] -> b
- (E) (b -> a -> b) -> b -> [a] -> b

http://tiny.cc/cse116-foldtype-grp

The “fold-right” pattern

Is `foldr` **tail recursive**?

Answer: No! It calls the binary operations on the results of the recursive call

What about tail-recursive versions?

Let's write tail-recursive `sum`!

```
sumTR :: [Int] -> Int
sumTR = ...
```

What about tail-recursive versions?

Let's write tail-recursive `sum`!

```
sumTR :: [Int] -> Int
sumTR xs = helper 0 xs
  where
    helper acc []     = acc
    helper acc (x:xs) = helper (acc + x) xs
```

What about tail-recursive versions?

Lets run `sumTR` to see how it works

```
sumTR [1,2,3]
  ==> helper 0 [1,2,3]
  ==> helper 1   [2,3]     -- 0 + 1 ==> 1
  ==> helper 3     [3]     -- 1 + 2 ==> 3
  ==> helper 6      []     -- 3 + 3 ==> 6
  ==> 6
```

Note: `helper` directly returns the result of recursive call!

What about tail-recursive versions?

Let's write tail-recursive `cat`!

```
catTR :: [String] -> String
catTR = ...
```

What about tail-recursive versions?

Let's write tail-recursive `cat`!

```
catTR :: [String] -> String
catTR xs = helper "" xs
  where
    helper acc []     = acc
    helper acc (x:xs) = helper (acc ++ x) xs
```

What about tail-recursive versions?

Lets run `catTR` to see how it works

```
catTR                       ["carne", "asada", "torta"]
   ==> helper ""            ["carne", "asada", "torta"]
   ==> helper "carne"                ["asada", "torta"]
   ==> helper "carneasada"                    ["torta"]
   ==> helper "carneasadatorta"                      []
   ==> "carneasadatorta"
```

Note: `helper` directly returns the result of recursive call!

Can you spot the pattern?

```haskell
-- sumTR
foo xs                 = helper 0 xs
  where
    helper acc []      = acc
    helper acc (x:xs) = helper (acc + x) xs



-- catTR
foo xs                 = helper "" xs
  where
    helper acc []      = acc
    helper acc (x:xs) = helper (acc ++ x) xs

pattern = ...
```

The “fold-left” pattern

```
sum xs                  = helper 0 xs
  where
    helper acc []       = acc
    helper acc (x:xs) = helper (acc + x) xs
```

```
cat xs                  = helper “” xs
  where
    helper acc []       = acc
    helper acc (x:xs) = helper (acc ++ x) xs
```

```
foldl f b xs            = helper b xs
  where
    helper acc []       = acc
    helper acc (x:xs) = helper (f acc x) xs
```

The `foldl` Pattern

General Pattern

- Use a helper function with an extra accumulator argument
- To compute new accumulator, combine current accumulator with the head using some binary operation

The "fold-left" pattern

```
foldl f b xs             = helper b xs
  where
    helper acc []        = acc
    helper acc (x:xs)    = helper (f acc x) xs
```

Let's refactor `sumTR` and `catTR`:

```
sumTR = foldl ...  ...
```

```
catTR = foldl ...  ...
```

Factor the tail-recursion out!

QUIZ

What does this evaluate to? *

```
foldl f b xs            = helper b xs
  where
    helper acc []       = acc
    helper acc (x:xs) = helper (f acc x) xs

quiz = foldl (:) [] [1,2,3]
```

- (A) Type error
- (B) [1,2,3]
- (C) [3,2,1]
- (D) [[3],[2],[1]]
- (E) [[1],[2],[3]]

http://tiny.cc/cse116-foldl-ind

QUIZ

What does this evaluate to? *

```
foldl f b xs           = helper b xs
  where
    helper acc []      = acc
    helper acc (x:xs) = helper (f acc x) xs
```

```
quiz = foldl (:) [] [1,2,3]
```

- (A) Type error
- (B) [1,2,3]
- (C) [3,2,1]
- (D) [[3],[2],[1]]
- (E) [[1],[2],[3]]

http://tiny.cc/cse116-foldl-grp

QUIZ

What does this evaluate to? *

```
foldl f b xs             = helper b xs
  where
    helper acc []        = acc
    helper acc (x:xs) = helper (f acc x) xs
```

```
quiz = foldl (\xs x -> x : xs) [] [1,2,3]
```

- (A) Type error
- (B) [1,2,3]
- (C) [3,2,1]
- (D) [[3],[2],[1]]
- (E) [[1],[2],[3]]

http://tiny.cc/cse116-foldl2-ind

QUIZ

What does this evaluate to? *

```
foldl f b xs            = helper b xs
  where
    helper acc []       = acc
    helper acc (x:xs) = helper (f acc x) xs
```

```
quiz = foldl (\xs x -> x : xs) [] [1,2,3]
```

- (A) Type error
- (B) [1,2,3]
- (C) [3,2,1]
- (D) [[3],[2],[1]]
- (E) [[1],[2],[3]]

http://tiny.cc/cse116-foldl2-grp

The “fold-left” pattern

```
foldl f b                               [x1, x2, x3, x4]
  ==> helper b                          [x1, x2, x3, x4]
  ==> helper (f b x1)                       [x2, x3, x4]
  ==> helper (f (f b x1) x2)                    [x3, x4]
  ==> helper (f (f (f b x1) x2) x3)                 [x4]
  ==> helper (f (f (f (f b x1) x2) x3) x4)  []
  ==> (f (f (f (f b x1) x2) x3) x4)
```

Accumulate the values from the **left**

For example:

```
foldl (+) 0                             [1, 2, 3, 4]
  ==> helper 0                          [1, 2, 3, 4]
  ==> helper (0 + 1)                       [2, 3, 4]
  ==> helper ((0 + 1) + 2)                    [3, 4]
  ==> helper (((0 + 1) + 2) + 3)                 [4]
  ==> helper ((((0 + 1) + 2) + 3) + 4)  []
  ==> ((((0 + 1) + 2) + 3) + 4)
```

Left vs. Right

```
foldl f b [x1, x2, x3]  ==> f (f (f b x1) x2) x3 -- Left

foldr f b [x1, x2, x3]  ==> f x1 (f x2 (f x3 b)) -- Right
```

For example:

```
foldl (+) 0 [1, 2, 3]  ==> ((0 + 1) + 2) + 3  -- Left

foldr (+) 0 [1, 2, 3]  ==> 1 + (2 + (3 + 0))  -- Right
```

Different types!

```
foldl :: (b -> a -> b) -> b -> [a] -> b  -- Left

foldr :: (a -> b -> b) -> b -> [a] -> b  -- Right
```

Useful HOF: flip

```haskell
-- you can write
foldl (\xs x -> x : xs) [] [1,2,3]

-- more concisely like so:
foldl (flip (:))          [] [1,2,3]
```

What is the type of `flip`?

```haskell
flip :: (a -> b -> c) -> b -> a -> c
```

Useful HOF: compose

```
-- you can write
map (\x -> f (g x)) ys

-- more concisely like so:
map (f . g) ys
```

What is the type of `(.)`?

```
(.) :: (b -> c) -> (a -> b) -> a -> c
```

Higher Order Functions

Iteration patterns over collections:

- **Filter** values in a collection given a *predicate*
- **Map** (iterate) a given *transformation* over a collection
- **Fold** (reduce) a collection into a value, given a *binary operation* to combine results

Useful helper HOFs:

- **Flip** the order of function's (first two) arguments
- **Compose** two functions

Higher Order Functions

HOFs can be put into libraries to enable modularity

- Data structure **library** implements `map`, `filter`, `fold` for its collections
    - generic efficient implementation
    - generic optimizations: `map f (map g xs) --> map (f.g) xs`
- Data structure **clients** use HOFs with specific operations
    - no need to know the implementation of the collection

Enabled the “big data” revolution e.g. *MapReduce*, *Spark*

That’s all folks!