

CSE 114A

Foundations of Programming Languages

Higher-Order Functions

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Based on course materials developed by Nadia Polikarpova

Plan for this week

Last week:

- user-defined *data types*
 - and how to manipulate them using *pattern matching* and *recursion*
- how to make recursive functions more efficient with *tail recursion*

This week:

- code reuse with *higher-order functions* (HOFs)
- some useful HOFs: `map`, `filter`, and `fold`

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Recursion is good

- Recursive code mirrors recursive data
 - Base constructor -> Base case
 - Inductive constructor -> Inductive case (with recursive call)
- But it can get kinda repetitive!

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Example: evens

Let's write a function evens:

```
-- evens [] ==> []
-- evens [1,2,3,4] ==> [2,4]
evens :: [Int] -> [Int]
evens [] = ...
evens (x:xs) = ...
```

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Example: four-letter words

Let's write a function fourChars:

```
-- fourChars [] ==> []
-- fourChars ["i","must","do","work"] ==> ["must","work"]
fourChars :: [String] -> [String]
fourChars [] = ...
fourChars (x:xs) = ...
```

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Yikes, Most Code is the Same!

```
foo [] = []
foo (x:xs)
  | x mod 2 == 0 = x : foo xs
  | otherwise   =   foo xs
```

```
foo [] = []
foo (x:xs)
  | length x == 4 = x : foo xs
  | otherwise     =   foo xs
```

Only difference is **condition**

- `x mod 2 == 0` vs `length x == 4`

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Moral of the day

D.R.Y. Don't Repeat Yourself!

Can we

- *reuse* the general pattern and
- *substitute in* the custom condition?

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HOFs to the rescue!

General Pattern

- expressed as a *higher-order function*
- takes customizable operations as *arguments*

Specific Operation

- passed in as an argument to the HOF

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The “filter” pattern

```
evens [] = []
evens (x:xs)
  | x `mod` 2 == 0 = x : evens xs
  | otherwise    = evens xs

fourChars [] = []
fourChars (x:xs)
  | length x == 4 = x : fourChars xs
  | otherwise    = fourChars xs
```

```
filter f [] = []
filter f (x:xs)
  | f x = x : filter f xs
  | otherwise = filter f xs
```

Use the **filter** pattern
to avoid duplicating code!

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The “filter” pattern

General Pattern

- HOF filter
- Recursively traverse list and pick out elements that satisfy a predicate

Specific Operation

- Predicates `isEven` and `isFour`

```
filter f [] = []
filter f (x:xs)
  | f x = x : filter f xs
  | otherwise = filter f xs
```

```
evens = filter isEven
  where
    isEven x = x `mod` 2 == 0
```

```
fourChars = filter isFour
  where
    isFour x = length x == 4
```

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Let’s talk about types

```
-- evens [1,2,3,4] ==> [2,4]
evens :: [Int] -> [Int]
evens xs = filter isEven xs
  where
    isEven :: Int -> Bool
    isEven x = x `mod` 2 == 0
filter :: ???
```

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Let’s talk about types

```
-- evens [1,2,3,4] ==> [2,4]
evens :: [Int] -> [Int]
evens xs = filter isEven xs
  where
    isEven :: Int -> Bool
    isEven x = x `mod` 2 == 0
filter :: ???
```

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Let's talk about types

```
-- fourChars ["i","must","do","work"] ==> ["must","work"]
fourChars :: [String] -> [String]
fourChars xs = filter isFour xs
  where
    isFour :: String -> Bool
    isFour x = length x == 4
filter :: ???
```

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Let's talk about types

Uh oh! So what's the type of filter?

```
filter :: (Int -> Bool) -> [Int] -> [Int] -- ???
filter :: (String -> Bool) -> [String] -> [String] -- ???
```

- It *does not care* what the list elements are
 - as long as the predicate can handle them
- It's type is **polymorphic** (generic) in the type of list elements

```
-- For any type `a`
-- if you give me a predicate on `a`s
-- and a list of `a`s,
-- I'll give you back a list of `a`s
filter :: (a -> Bool) -> [a] -> [a]
```

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Example: all caps

Lets write a function shout:

```
-- shout [] ==> []
-- shout ['h','e','l','l','o'] ==> ['H','E','L','L','O']
shout :: [Char] -> [Char]
shout [] = ...
shout (x:xs) = ...
```

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Example: squares

Lets write a function squares:

```
-- squares []      ==> []
-- squares [1,2,3,4] ==> [1,4,9,16]
squares :: [Int] -> [Int]
squares []      = ...
squares (x:xs) = ...
```

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Yikes, Most Code is the Same!

Lets rename the functions to foo:

```
-- shout
foo []      = []
foo (x:xs) = toUpper x : foo xs

-- squares
foo []      = []
foo (x:xs) = (x * x) : foo xs
```

Lets refactor into the common pattern

```
pattern = ...
```

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The “map” pattern

<pre>shout [] = [] shout (x:xs) = toUpper x : shout xs</pre>	<pre>squares [] = [] squares (x:xs) = (x*x) : squares xs</pre>
---	---

<pre>map f [] = [] map f (x:xs) = f x : map f xs</pre>

The map Pattern

General Pattern

- HOF map
- Apply a transformation f to each element of a list

Specific Operations

- Transformations toUpper and \x -> x * x

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The “map” pattern

```
map f [] = []  
map f (x:xs) = f x : map f xs
```

Lets refactor shout and squares

```
shout = map ...
```

```
squares = map ...
```

```
map f [] = []  
map f (x:xs) = f x : map f xs
```

```
shout = map (\x -> toUpper x)  
squares = map (\x -> x*x)
```

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QUIZ

What is the type of map? *

```
map f [] = []  
map f (x:xs) = f x : map f xs
```

- (A) `(Char -> Char) -> [Char] -> [Char]`
- (B) `(Int -> Int) -> [Int] -> [Int]`
- (C) `(a -> a) -> [a] -> [a]`
- (D) `(a -> b) -> [a] -> [b]`
- (E) `(a -> b) -> [c] -> [d]`



<http://tiny.cc/cse116-map-ind>

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QUIZ

What is the type of map? *

```
map f [] = []  
map f (x:xs) = f x : map f xs
```

- (A) `(Char -> Char) -> [Char] -> [Char]`
- (B) `(Int -> Int) -> [Int] -> [Int]`
- (C) `(a -> a) -> [a] -> [a]`
- (D) `(a -> b) -> [a] -> [b]`
- (E) `(a -> b) -> [c] -> [d]`



<http://tiny.cc/cse116-map-grp>

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The “map” pattern

```
-- For any types `a` and `b`
-- if you give me a transformation from `a` to `b`
-- and a list of `a`s,
-- I'll give you back a List of `b`s
map :: (a -> b) -> [a] -> [b]
```

Type says it all!

- The only meaningful thing a function of this type can do is apply its first argument to elements of the list (Hoogle it!)

Things to try at home:

- can you write a function `map' :: (a -> b) -> [a] -> [b]` whose behavior is different from `map`?
- can you write a function `map' :: (a -> b) -> [a] -> [b]` such that `map' f xs` returns a list whose elements are not in `map f xs`?

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QUIZ

What is the value of quiz? *

```
map :: (a -> b) -> [a] -> [b]
```

```
quiz = map (\(x, y) -> x + y) [1, 2, 3]
```

- (A) [2, 4, 6]
- (B) [3, 5]
- (C) Syntax Error
- (D) Type Error
- (E) None of the above



<http://tiny.cc/cse116-quiz-ind>

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QUIZ

What is the value of quiz? *

```
map :: (a -> b) -> [a] -> [b]
```

```
quiz = map (\(x, y) -> x + y) [1, 2, 3]
```

- (A) [2, 4, 6]
- (B) [3, 5]
- (C) Syntax Error
- (D) Type Error
- (E) None of the above



<http://tiny.cc/cse116-quiz-grp>

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Don't Repeat Yourself

Benefits of **factoring** code with HOFs:

- Reuse iteration pattern
 - think in terms of standard patterns
 - less to write
 - easier to communicate
- Avoid bugs due to repetition

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Recall: length of a list

```
-- len [] ==> 0
-- len ["carne","asada"] ==> 2
len :: [a] -> Int
len [] = 0
len (x:xs) = 1 + len xs
```

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Recall: summing a list

```
-- sum [] ==> 0
-- sum [1,2,3] ==> 6
sum :: [Int] -> Int
sum [] = 0
sum (x:xs) = x + sum xs
```

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Example: string concatenation

Let's write a function `cat`:

```
-- cat [] ==> ""
-- cat ["carne", "asada", "torta"] ==> "carneasadatorta"
cat :: [String] -> String
cat [] = ...
cat (x:xs) = ...
```

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Can you spot the pattern?

```
-- len
foo [] = 0
foo (x:xs) = 1 + foo xs

-- sum
foo [] = 0
foo (x:xs) = x + foo xs

-- cat
foo [] = ""
foo (x:xs) = x ++ foo xs

pattern = ...
```

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The “fold-right” pattern

<pre>len [] = 0 len (x:xs) = 1 + len xs</pre>	<pre>sum [] = 0 sum (x:xs) = x + sum xs</pre>	<pre>cat [] = "" cat (x:xs) = x ++ sum xs</pre>
---	---	---

<pre>foldr f b [] = b foldr f b (x:xs) = f x (foldr f b xs)</pre>

The foldr Pattern

General Pattern

- Recurse on tail
- Combine result with the head using some binary operation

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The “fold-right” pattern

```
foldr f b [] = b
foldr f b (x:xs) = f x (foldr f b xs)
```

Let's refactor sum, len and cat:

```
sum = foldr ... ..
```

```
cat = foldr ... ..
```

```
len = foldr ... ..
```

Factor the recursion out!

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The “fold-right” pattern

```
foldr f b [] = b
foldr f b (x:xs) = f x (foldr f b xs)
```

```
len = foldr (\x n -> 1 + n) 0
```

```
sum = foldr (\x n -> x + n) 0
```

```
cat = foldr (\x s -> x ++ s) ""
```

You can write it more clearly as

```
sum = foldr (+) 0
```

```
cat = foldr (++) ""
```

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The “fold-right” pattern

```
foldr f b [] = b
foldr f b (x:xs) = f x (foldr f b xs)
```

```
len = foldr (\x n -> 1 + n) 0
```

```
sum = foldr (\x n -> x + n) 0
```

```
cat = foldr (\x s -> x ++ s) ""
```

You can write it more clearly as

```
sum = foldr (+) 0
```

```
cat = foldr (++) ""
```

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QUIZ

What does this evaluate to? *

```
foldr f b [] = b
foldr f b (x:xs) = f x (foldr f b xs)
```

```
quiz = foldr (:) [] [1,2,3]
```

- (A) Type error
- (B) [1,2,3]
- (C) [3,2,1]
- (D) [[3],[2],[1]]
- (E) [[1],[2],[3]]



<http://tiny.cc/cse116-foldeval-ind>

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QUIZ

What does this evaluate to? *

```
foldr f b [] = b
foldr f b (x:xs) = f x (foldr f b xs)
```

```
quiz = foldr (:) [] [1,2,3]
```

- (A) Type error
- (B) [1,2,3]
- (C) [3,2,1]
- (D) [[3],[2],[1]]
- (E) [[1],[2],[3]]



<http://tiny.cc/cse116-foldeval-grp>

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The “fold-right” pattern

```
foldr f b [] = b
foldr f b (x:xs) = f x (foldr f b xs)
```

```
foldr (:) [] [1,2,3]
==> (:) 1 (foldr (:) [] [2, 3])
==> (:) 1 ((:) 2 (foldr (:) [] [3]))
==> (:) 1 ((:) 2 ((:) 3 (foldr (:) [] [])))
==> (:) 1 ((:) 2 ((:) 3 []))
== 1 : (2 : (3 : []))
== [1,2,3]
```

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The “fold-right” pattern

```
foldr f b [x1, x2, x3, x4]
==> f x1 (foldr f b [x2, x3, x4])
==> f x1 (f x2 (foldr f b [x3, x4]))
==> f x1 (f x2 (f x3 (foldr f b [x4])))
==> f x1 (f x2 (f x3 (f x4 (foldr f b []))))
==> f x1 (f x2 (f x3 (f x4 b)))
```

Accumulate the values from the **right**

For example:

```
foldr (+) 0 [1, 2, 3, 4]
==> 1 + (foldr (+) 1 [2, 3, 4])
==> 1 + (2 + (foldr (+) 0 [3, 4]))
==> 1 + (2 + (3 + (foldr (+) 0 [4])))
==> 1 + (2 + (3 + (4 + (foldr (+) 0 []))))
==> 1 + (2 + (3 + (4 + 0)))
```

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QUIZ

What is the most general type of foldr? *

```
foldr f b [] = b
foldr f b (x:xs) = f x (foldr f b xs)
```

- (A) $(a \rightarrow a \rightarrow a) \rightarrow a \rightarrow [a] \rightarrow a$
- (B) $(a \rightarrow a \rightarrow b) \rightarrow a \rightarrow [a] \rightarrow b$
- (C) $(a \rightarrow b \rightarrow a) \rightarrow b \rightarrow [a] \rightarrow b$
- (D) $(a \rightarrow b \rightarrow b) \rightarrow b \rightarrow [a] \rightarrow b$
- (E) $(b \rightarrow a \rightarrow b) \rightarrow b \rightarrow [a] \rightarrow b$



<http://tiny.cc/cse116-foldtype-ind>

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QUIZ

What is the most general type of foldr? *

```
foldr f b [] = b
foldr f b (x:xs) = f x (foldr f b xs)
```

- (A) $(a \rightarrow a \rightarrow a) \rightarrow a \rightarrow [a] \rightarrow a$
- (B) $(a \rightarrow a \rightarrow b) \rightarrow a \rightarrow [a] \rightarrow b$
- (C) $(a \rightarrow b \rightarrow a) \rightarrow b \rightarrow [a] \rightarrow b$
- (D) $(a \rightarrow b \rightarrow b) \rightarrow b \rightarrow [a] \rightarrow b$
- (E) $(b \rightarrow a \rightarrow b) \rightarrow b \rightarrow [a] \rightarrow b$



<http://tiny.cc/cse116-foldtype-grp>

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The “fold-right” pattern

Is `foldr` tail recursive?

Answer: No! It calls the binary operations on the results of the recursive call

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What about tail-recursive versions?

Let's write tail-recursive sum!

```
sumTR :: [Int] -> Int
sumTR = ...
```

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What about tail-recursive versions?

Let's write tail-recursive sum!

```
sumTR :: [Int] -> Int
sumTR xs = helper 0 xs
  where
    helper acc [] = acc
    helper acc (x:xs) = helper (acc + x) xs
```

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What about tail-recursive versions?

Lets run sumTR to see how it works

```
sumTR [1,2,3]
==> helper 0 [1,2,3]
==> helper 1 [2,3] -- 0 + 1 ==> 1
==> helper 3 [3] -- 1 + 2 ==> 3
==> helper 6 [] -- 3 + 3 ==> 6
==> 6
```

Note: helper directly returns the result of recursive call!

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What about tail-recursive versions?

Let's write tail-recursive cat!

```
catTR :: [String] -> String
catTR = ...
```

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What about tail-recursive versions?

Let's write tail-recursive cat!

```
catTR :: [String] -> String
catTR xs = helper "" xs
  where
    helper acc [] = acc
    helper acc (x:xs) = helper (acc ++ x) xs
```

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What about tail-recursive versions?

Lets run `catTR` to see how it works

```
catTR ["carne", "asada", "torta"]
==> helper "" ["carne", "asada", "torta"]
==> helper "carne" ["asada", "torta"]
==> helper "carneasada" ["torta"]
==> helper "carneasadatorta" []
==> "carneasadatorta"
```

Note: `helper` directly returns the result of recursive call!

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Can you spot the pattern?

```
-- sumTR
foo xs = helper 0 xs
where
  helper acc [] = acc
  helper acc (x:xs) = helper (acc + x) xs

-- catTR
foo xs = helper "" xs
where
  helper acc [] = acc
  helper acc (x:xs) = helper (acc ++ x) xs

pattern = ...
```

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The “fold-left” pattern

<pre>sum xs = helper 0 xs where helper acc [] = acc helper acc (x:xs) = helper (acc + x) xs</pre>	<pre>cat xs = helper "" xs where helper acc [] = acc helper acc (x:xs) = helper (acc ++ x) xs</pre>
---	---

<pre>foldl f b xs = helper b xs where helper acc [] = acc helper acc (x:xs) = helper (f acc x) xs</pre>

The `foldl` Pattern

General Pattern

- Use a helper function with an extra accumulator argument
- To compute new accumulator, combine current accumulator with the head using some binary operation

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The “fold-left” pattern

```
foldl f b xs      = helper b xs
  where
    helper acc [] = acc
    helper acc (x:xs) = helper (f acc x) xs
```

Let's refactor sumTR and catTR:

```
sumTR = foldl ... ..
```

```
catTR = foldl ... ..
```

Factor the tail-recursion out!

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QUIZ

What does this evaluate to? *

```
foldl f b xs      = helper b xs
  where
    helper acc [] = acc
    helper acc (x:xs) = helper (f acc x) xs
```

```
quiz = foldl (:) [] [1,2,3]
```

- (A) Type error
- (B) [1,2,3]
- (C) [3,2,1]
- (D) [[3],[2],[1]]
- (E) [[1],[2],[3]]



<http://tiny.cc/cse116-foldl-ind>

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QUIZ

What does this evaluate to? *

```
foldl f b xs      = helper b xs
  where
    helper acc [] = acc
    helper acc (x:xs) = helper (f acc x) xs
```

```
quiz = foldl (:) [] [1,2,3]
```

- (A) Type error
- (B) [1,2,3]
- (C) [3,2,1]
- (D) [[3],[2],[1]]
- (E) [[1],[2],[3]]



<http://tiny.cc/cse116-foldl-grp>

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QUIZ

What does this evaluate to? *

```
foldl f b xs      = helper b xs
  where
    helper acc [] = acc
    helper acc (x:xs) = helper (f acc x) xs
```

```
quiz = foldl (\xs x -> x : xs) [] [1,2,3]
```

- (A) Type error
- (B) [1,2,3]
- (C) [3,2,1]
- (D) [[3],[2],[1]]
- (E) [[1],[2],[3]]



<http://tiny.cc/cse116-foldl2-ind>

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QUIZ

What does this evaluate to? *

```
foldl f b xs      = helper b xs
  where
    helper acc [] = acc
    helper acc (x:xs) = helper (f acc x) xs
```

```
quiz = foldl (\xs x -> x : xs) [] [1,2,3]
```

- (A) Type error
- (B) [1,2,3]
- (C) [3,2,1]
- (D) [[3],[2],[1]]
- (E) [[1],[2],[3]]



<http://tiny.cc/cse116-foldl2-grp>

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The “fold-left” pattern

```
foldl f b [x1, x2, x3, x4]
==> helper b [x1, x2, x3, x4]
==> helper (f b x1) [x2, x3, x4]
==> helper (f (f b x1) x2) [x3, x4]
==> helper (f (f (f b x1) x2) x3) [x4]
==> helper (f (f (f (f b x1) x2) x3) x4) []
==> (f (f (f (f b x1) x2) x3) x4)
```

Accumulate the values from the left

For example:

```
foldl (+) 0 [1, 2, 3, 4]
==> helper 0 [1, 2, 3, 4]
==> helper (0 + 1) [2, 3, 4]
==> helper ((0 + 1) + 2) [3, 4]
==> helper (((0 + 1) + 2) + 3) [4]
==> helper ((((0 + 1) + 2) + 3) + 4) []
==> (((((0 + 1) + 2) + 3) + 4)
```

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Left vs. Right

```
foldl f b [x1, x2, x3] ==> f (f (f b x1) x2) x3 -- Left
```

```
foldr f b [x1, x2, x3] ==> f x1 (f x2 (f x3 b)) -- Right
```

For example:

```
foldl (+) 0 [1, 2, 3] ==> ((0 + 1) + 2) + 3 -- Left
```

```
foldr (+) 0 [1, 2, 3] ==> 1 + (2 + (3 + 0)) -- Right
```

Different types!

```
foldl :: (b -> a -> b) -> b -> [a] -> b -- Left
```

```
foldr :: (a -> b -> b) -> b -> [a] -> b -- Right
```

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Useful HOF: flip

-- you can write

```
foldl (\xs x -> x : xs) [] [1,2,3]
```

-- more concisely like so:

```
foldl (flip (:)) [] [1,2,3]
```

What is the type of flip?

```
flip :: (a -> b -> c) -> b -> a -> c
```

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Useful HOF: compose

-- you can write

```
map (\x -> f (g x)) ys
```

-- more concisely like so:

```
map (f . g) ys
```

What is the type of (.)?

```
(.) :: (b -> c) -> (a -> b) -> a -> c
```

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Higher Order Functions

Iteration patterns over collections:

- **Filter** values in a collection given a *predicate*
- **Map** (iterate) a given *transformation* over a collection
- **Fold** (reduce) a collection into a value, given a *binary operation* to combine results

Useful helper HOFs:

- **Flip** the order of function's (first two) arguments
- **Compose** two functions

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Higher Order Functions

HOFs can be put into libraries to enable modularity

- Data structure **library** implements `map`, `filter`, `fold` for its collections
 - generic efficient implementation
 - generic optimizations: `map f (map g xs) --> map (f.g) xs`
- Data structure **clients** use HOFs with specific operations
 - no need to know the implementation of the collection

Enabled the “big data” revolution e.g. *MapReduce*, *Spark*

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That's all folks!

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