

CSE 114A: Fall 2023

**Foundations of Programming
Languages**

Lambda Calculus

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Based on course materials developed by Ranjit Jhala

Your favorite language

- Probably has lots of features:
 - Assignment ($x = x + 1$)
 - Booleans, integers, characters, strings,...
 - Conditionals
 - Loops, return, break, continue
 - Functions
 - Recursion
 - References / pointers
 - Objects and classes
 - Inheritance
 - ... and more

Your favorite language

- Probably has lots of features:

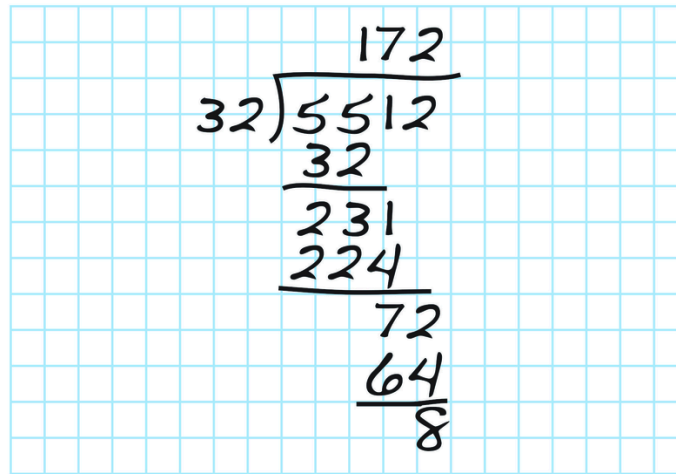
- Assignment ($x = x + 1$)
- Booleans, integers, characters, strings, ...
- Conditionals
- Loops return break continue
- []
- []
- References / pointers
- Objects and classes
- Inheritance
- ... and more

Which ones can we do without?

What is the smallest universal language?

What is computable?

- Prior to 1930s
 - Informal notion of an effectively calculable function:



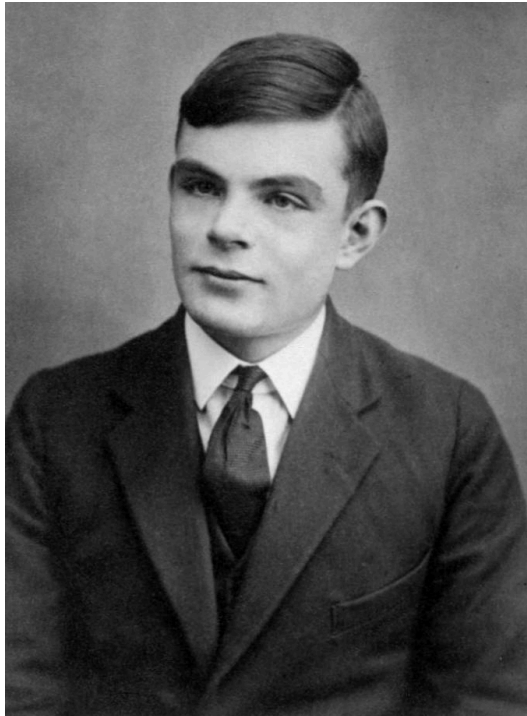
A handwritten long division problem on a light blue grid background. The problem is $32 \overline{)5512}$. The quotient is 172. The steps are: 32 goes into 55 one time (1), leaving a remainder of 23. Bring down the next digit (1) to get 231. 32 goes into 231 seven times (7), leaving a remainder of 72. Bring down the final digit (2) to get 72. 32 goes into 72 two times (2), leaving a remainder of 8. The final result is 172 with a remainder of 8.

$$\begin{array}{r} 172 \\ 32 \overline{)5512} \\ \underline{32} \\ 231 \\ \underline{224} \\ 72 \\ \underline{64} \\ 8 \end{array}$$

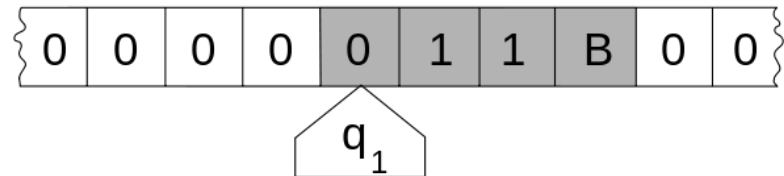
One that can be computed by a human with pen and paper, following an algorithm

What is computable?

- 1936: Formalization



Alan Turing: Turing machines



What is computable?

- 1936: Formalization



Alonzo Church: lambda calculus

$$e ::= x$$
$$| \lambda x \rightarrow e$$
$$| e_1 e_2$$

The Next 700 Languages

- Big impact on language design!



Whatever the next 700 languages turn out to be, they will surely be variants of lambda calculus.

Peter Landin, 1966

Your favorite language

- Probably has lots of features:
 - Assignment ($x = x + 1$)
 - Booleans, integers, characters, strings,...
 - Conditionals
 - Loops, return, break, continue
 - Functions
 - Recursion
 - References / pointers
 - Objects and classes
 - Inheritance
 - ... and more

The Lambda Calculus

- Features
 - Functions
 - (that's it)

The Lambda Calculus

- Seriously...
 - ~~Assignment ($x = x + 1$)~~
 - ~~Booleans, integers, characters, strings,...~~
 - ~~Conditionals~~
 - ~~Loops, return, break, continue~~
 - **Functions**
 - ~~Recursion~~
 - ~~References / pointers~~
 - ~~Objects and classes~~
 - ~~Inheritance~~
 - ~~... and more~~

The only thing you can do is:
Define a function
Call a function

Describing a Programming Language

- **Syntax**
 - What do programs *look like*?
- **Semantics**
 - What do programs *mean*?
 - **Operational semantics:**
 - How do programs *execute step-by-step*?

Syntax: What programs look like

$$e ::= x$$
$$| \lambda x \rightarrow e$$
$$| e1 e2$$

- Programs are *expressions* e (also called λ -terms)
- **Variable:** x, y, z
- **Abstraction** (aka nameless function definition):
 - $\lambda x \rightarrow e$ “for any x , compute e ”
 - x is the *formal parameter*, e is the *body*
- **Application** (aka function call):
 - $e1 e2$ “apply $e1$ to $e2$ ”
 - $e1$ is the *function*, $e2$ is the *argument*

Examples

-- The identity function ("for any x compute x")

```
\x -> x
```

-- A function that returns the identity function

```
\x -> (\y -> y)
```

-- A function that applies its argument to

-- the identity function

```
\f -> f (\x -> x)
```

QUIZ: Lambda syntax

Which of the following terms are syntactically incorrect? *

- A. $\lambda(x \rightarrow x) \rightarrow y$
- B. $\lambda x \rightarrow x x$
- C. $\lambda x \rightarrow x (y x)$
- A and C
- All of the above



<http://tiny.cc/cse116-lambda-ind>

QUIZ: Lambda syntax

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<http://tiny.cc/cse116-lambda-grp>

Examples

-- The identity function ("for any x compute x")

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-- A function that returns the identity function

```
\x -> (\y -> y)
```

-- A function that applies its argument to

-- the identity function

```
\f -> f (\x -> x)
```

- How do I define a function with two arguments?
 - e.g. a function that takes x and y and returns y

Examples

-- A function that returns the identity function

```
\x -> (\y -> y)
```

OR: a function that takes two arguments and returns the second one!

- How do I define a function with two arguments?
 - e.g. a function that takes x and y and returns y

Examples

- How do I apply a function to two arguments?
 - e.g. apply `\x -> (\y -> y)` to apple and banana?

-- first apply to apple, then apply the result to banana

```
(((\x -> (\y -> y)) apple) banana)
```

Syntactic Sugar

- Convenient notation used as a shorthand for valid syntax

instead of	we write
<code>\x -> (\y -> (\z -> e))</code>	<code>\x -> \y -> \z -> e</code>
<code>\x -> \y -> \z -> e</code>	<code>\x y z -> e</code>
<code>((e1 e2) e3) e4</code>	<code>e1 e2 e3 e4</code>

`\x y -> y` *-- A function that that takes two arguments
-- and returns the second one...*

`(\x y -> y) apple banana` *-- ... applied to two arguments*

Semantics: What programs mean

- How do I “run” or “execute” a λ -term?
- Think of middle-school algebra:

-- *Simplify expression:*

$$(x + 2) * (3 * x - 1)$$

=

???

- **Execute** = rewrite step-by-step following simple rules until no more rules apply

Rewrite rules of lambda calculus

1. α -step (aka renaming formals)
2. β -step (aka function call)

But first we have to talk about **scope**

Semantics: Scope of a Variable

- The part of a program where a **variable is visible**
- In the expression $\lambda x \rightarrow e$
 - x is the newly introduced variable
 - e is the **scope** of x
 - any **occurrence** of x in $\lambda x \rightarrow e$ is **bound** (by the **binder** λx)

Semantics: Scope of a Variable

- For example, x is **bound** in:

$\lambda x \rightarrow x$

$\lambda x \rightarrow (\lambda y \rightarrow x)$

- An occurrence of x in e is **free** if it's *not bound* by an enclosing abstraction

- For example, x is **free** in:

$x y$ -- *no binders at all!*

$\lambda y \rightarrow x y$ -- *no λx binder*

$(\lambda x \rightarrow \lambda y \rightarrow y) x$ -- *x is outside the scope
-- of the λx binder;*

-- *intuition: it's not "the same" x*

QUIZ: Variable scope

In the expression $(\lambda x \rightarrow x) x$, is x bound or free? *

- A. bound
- B. free
- C. first occurrence is bound, second is free
- D. first occurrence is bound, second and third are free
- E. first two occurrences are bound, third is free



<http://tiny.cc/cse116-scope-ind>

QUIZ: Variable scope

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<http://tiny.cc/cse116-scope-grp>

Free Variables

- An variable x is **free** in e if there exists a free occurrence of x in e
- We can formally define the set of all free variables in a term like so:

$$FV(x) = ???$$

$$FV(\lambda x \rightarrow e) = ???$$

$$FV(e1 \ e2) = ???$$

Free Variables

- An variable x is **free** in e if there exists a free occurrence of x in e
- We can formally define the set of all free variables in a term like so:

$$FV(x) = \{x\}$$

$$FV(\lambda x \rightarrow e) = FV(e) \setminus \{x\}$$

$$FV(e_1 \ e_2) = FV(e_1) \cup FV(e_2)$$

Closed Expressions

- If e has no free variables it is said to be closed
- Closed expressions are also called **combinators**
 - **Q:** What is the *shortest* closed expression?

Closed Expressions

- If e has no free variables it is said to be closed
- Closed expressions are also called **combinators**
 - **Q:** What is the *shortest* closed expression?
 - **A:** $\lambda x . x$

Rewrite rules of lambda calculus

1. α -step (aka renaming formals)
2. β -step (aka function call)

Semantics: β -Reduction

$$(\lambda x \rightarrow e1) e2 \quad =_{\beta} \quad e1[x := e2]$$

where $e1[x := e2]$ means “ $e1$ with all free occurrences of x replaced with $e2$ ”

- Computation by *search-and-replace*:
 - If you see an *abstraction* applied to an argument, take the *body* of the abstraction and replace all free occurrences of the *formal* by that argument
 - We say that $(\lambda x \rightarrow e1) e2$ β -steps to $e1[x := e2]$

Examples

```
(\x -> x) apple  
=b> apple
```

Is this right? Ask [Elsa!](#)

```
(\f -> f (\x -> x)) (give apple)  
=b> ???
```


Examples

```
(\x -> x) apple  
=b> apple
```

Is this right? Ask [Elsa!](#)

```
(\f -> f (\x -> x)) (give apple)  
=b> give apple (\x -> x)
```

Elsa Demo

QUIZ: β -Reduction 1

$(\lambda x \rightarrow (\lambda y \rightarrow y)) \text{ apple} = b > ??? *$

- A. apple
- B. $\lambda y \rightarrow \text{apple}$
- C. $\lambda x \rightarrow \text{apple}$
- D. $\lambda y \rightarrow y$
- E. $\lambda x \rightarrow y$



<http://tiny.cc/cse116-beta1-ind>

QUIZ: β -Reduction 1

$(\lambda x \rightarrow (\lambda y \rightarrow y)) \text{ apple} = b \rightarrow ???$ \rightarrow

- A. apple
- B. $\lambda y \rightarrow \text{apple}$
- C. $\lambda x \rightarrow \text{apple}$
- D. $\lambda y \rightarrow y$
- E. $\lambda x \rightarrow y$



<http://tiny.cc/cse116-beta1-grp>

QUIZ: β -Reduction 2

$(\lambda x \rightarrow x (\lambda x \rightarrow x)) \text{ apple} = b > ??? *$

- A. $\text{apple } (\lambda x \rightarrow x)$
- B. $\text{apple } (\lambda \text{apple} \rightarrow \text{apple})$
- C. $\text{apple } (\lambda x \rightarrow \text{apple})$
- D. apple
- E. $\lambda x \rightarrow x$



<http://tiny.cc/cse116-beta2-ind>

QUIZ: β -Reduction 2

$(\lambda x \rightarrow x (\lambda x \rightarrow x)) \text{ apple} = b > ??? *$

- A. $\text{apple } (\lambda x \rightarrow x)$
- B. $\text{apple } (\lambda \text{apple} \rightarrow \text{apple})$
- C. $\text{apple } (\lambda x \rightarrow \text{apple})$
- D. apple
- E. $\lambda x \rightarrow x$



<http://tiny.cc/cse116-beta2-grp>

A Tricky One

```
(\x -> (\y -> x)) y  
=> \y -> y
```

Is this right?

Problem: the free y in the argument has been *captured* by $\backslash y$!

Solution: make sure that all *free variables* of the argument are different from the *binders* in the body.

Capture-Avoiding Substitution

- We have to fix our definition of β -reduction:

$$(\lambda x \rightarrow e1) e2 \quad =_{\beta} \quad e1[x := e2]$$

where $e1[x := e2]$ means “ ~~$e1$ with all free occurrences of x replaced with $e2$ ”~~”

- $e1$ with all *free* occurrences of x replaced with $e2$, **as long as** no free variables of $e2$ get captured
- undefined otherwise

Capture-Avoiding Substitution

Formally:

```
x[x := e]           = e
y[x := e]           = y    -- assuming x /= y
(e1 e2)[x := e]     = (e1[x := e]) (e2[x := e])
(\x -> e1)[x := e]  = \x -> e1 -- why just `e1`?

(\y -> e1)[x := e]
  | not (y in FV(e)) = \y -> e1[x := e]
  | otherwise        = undefined -- but what then???
```

Rewrite rules of lambda calculus

1. α -step (aka renaming formals)
2. β -step (aka function call)

Semantics: α -Reduction

$\lambda x \rightarrow e \quad =_a \quad \lambda y \rightarrow e[x := y]$
where not $(y \text{ in } FV(e))$

- We can rename a formal parameter and replace all its occurrences in the body
- We say that $(\lambda x \rightarrow e)$ *a-steps* to $(\lambda y \rightarrow e[x := y])$

Semantics: α -Reduction

$\lambda x \rightarrow e \quad =_{\alpha} \quad \lambda y \rightarrow e[x := y]$
where not (y in FV(e))

- Example:

$\lambda x \rightarrow x \quad =_{\alpha} \quad \lambda y \rightarrow y \quad =_{\alpha} \quad \lambda z \rightarrow z$

- All these expressions are α -equivalent

Example

What's wrong with these?

-- (A)

$\lambda f \rightarrow f\ x \quad =a> \quad \lambda x \rightarrow x\ x$

-- (B)

$(\lambda x \rightarrow \lambda y \rightarrow y)\ y \quad =a> \quad (\lambda x \rightarrow \lambda z \rightarrow z)\ z$

-- (C)

$\lambda x \rightarrow \lambda y \rightarrow x\ y \quad =a> \quad \lambda \text{apple} \rightarrow \lambda \text{orange} \rightarrow \text{apple}\ \text{orange}$

The Tricky One

$(\lambda x \rightarrow (\lambda y \rightarrow x)) y$
=a> ???

To avoid getting confused, you can always rename formals, so that different variables have different names!

The Tricky One

```
(\x -> (\y -> x)) y  
=a> (\x -> (\z -> x)) y  
=b> \z -> y
```

To avoid getting confused, you can always rename formals, so that different variables have different names!

Normal Forms

A **redex** is a λ -term of the form

$$(\lambda x \rightarrow e1) e2$$

A λ -term is in **normal form** if it contains no redexes.

QUIZ: Normal form

Which of the following terms are not in normal form ? *

- A. x
- B. $x y$
- C. $(\neg x \rightarrow x) y$
- D. $x (\neg y \rightarrow y)$
- E. C and D



<http://tiny.cc/cse116-norm-ind>

QUIZ: Normal form

Which of the following terms are not in normal form ? *

- A. x
- B. $x y$
- C. $(\neg x \rightarrow x) y$
- D. $x (\neg y \rightarrow y)$
- E. C and D



<http://tiny.cc/cse116-norm-grp>

Semantics: Evaluation

- A λ -term e evaluates to e' if

1. There is a sequence of steps

$$e \Rightarrow e_1 \Rightarrow \dots \Rightarrow e_N \Rightarrow e'$$

where each \Rightarrow is either \Rightarrow_a or \Rightarrow_b and $N \geq 0$

2. e' is in *normal form*

Example of evaluation

```
(\x -> x) apple  
=b> apple
```

```
(\f -> f (\x -> x)) (\x -> x)  
=?> ???
```

```
(\x -> x x) (\x -> x)  
=?> ???
```

Example of evaluation

```
(\x -> x) apple  
=b> apple
```

```
(\f -> f (\x -> x)) (\x -> x)  
=b> (\x -> x) (\x -> x)  
=b> \x -> x
```

```
(\x -> x x) (\x -> x)  
=?> ???
```

Example of evaluation

```
(\x -> x) apple  
=b> apple
```

```
(\f -> f (\x -> x)) (\x -> x)  
=b> (\x -> x) (\x -> x)  
=b> \x -> x
```

```
(\x -> x x) (\x -> x)  
=b> (\x -> x) (\x -> x)  
=b> \x -> x
```

Elsa shortcuts

- Named λ -terms

```
let ID = \x -> x  -- abbreviation for \x -> x
```

- To substitute a name with its definition, use a `=d>` step:

```
ID apple
=d> (\x -> x) apple  -- expand definition
=b> apple           -- beta-reduce
```

Elsa shortcuts

- Evaluation
 - $e1 \Rightarrow^* e2$: $e1$ reduces to $e2$ in 0 or more steps
 - where each step is \Rightarrow^a , \Rightarrow^b , or \Rightarrow^d
 - $e1 \Rightarrow^{\sim} e2$: $e1$ evaluates to $e2$
- *What is the difference?*

Non-Terminating Evaluation

$(\lambda x \rightarrow x x) (\lambda x \rightarrow x x)$
 $=b> (\lambda x \rightarrow x x) (\lambda x \rightarrow x x)$

- Oh no... we can write programs that loop back to themselves
- And never reduce to normal form!
- This combinator is called Ω

Non-Terminating Evaluation

- What if we pass Ω as an argument to another function?

```
let OMEGA = (\x -> x x) (\x -> x x)
```

```
(\x -> \y -> y) OMEGA
```

- Does this reduce to a normal form? Try it at home!

Programming in λ -calculus

- Real languages have lots of features
 - Booleans
 - Records (structs, tuples)
 - Numbers
 - **Functions** [we got those]
 - Recursion
- Let's see how to encode all of these features with the λ -calculus.

λ -calculus: Booleans

- How can we encode Boolean values (TRUE and FALSE) as functions?
- Well, what do we **do** with a Boolean **b**?

- We make a *binary choice*

if b **then** e1 **else** e2

Booleans: API

- We need to define three functions

```
let TRUE  = ???
```

```
let FALSE = ???
```

```
let ITE    = \b x y -> ???  -- if b then x else y
```

such that

```
ITE TRUE apple banana =~> apple
```

```
ITE FALSE apple banana =~> banana
```

(Here, `let NAME = e` means `NAME` is an *abbreviation* for `e`)

Booleans: Implementation

```
let TRUE  = \x y -> x      -- Returns first argument
let FALSE = \x y -> y      -- Returns second argument
let ITE   = \b x y -> b x y -- Applies cond. to branches
                                     -- (redundant, but
                                     -- improves readability)
```

Example: Branches step-by-step

eval ite_true:

ITE TRUE e1 e2

=d> ($\lambda b x y \rightarrow b \quad x \quad y$) TRUE e1 e2 -- *expand def ITE*

=b> ($\lambda x y \rightarrow$ TRUE $x \quad y$) e1 e2 -- *beta-step*

=b> ($\lambda y \rightarrow$ TRUE e1 y) e2 -- *beta-step*

=b> TRUE e1 e2 -- *expand def TRUE*

=d> ($\lambda x y \rightarrow x$) e1 e2 -- *beta-step*

=b> ($\lambda y \rightarrow e1$) e2 -- *beta-step*

=b> e1

Example: Branches step-by-step

- Now you try it!
- Can you fill in the blanks to make it happen?
 - <http://goto.ucsd.edu/elsa/index.html#?demo=ite.lc>

```
eval ite_false:
```

```
  ITE FALSE e1 e2
```

```
-- fill the steps in!
```

```
=b> e2
```


Example: Branches step-by-step

eval ite_false:

ITE FALSE e1 e2

=d> ($\lambda b x y \rightarrow b \quad x \quad y$) FALSE e1 e2 -- expand def ITE

=b> ($\lambda x y \rightarrow$ FALSE x y) e1 e2 -- beta-step

=b> ($\lambda y \rightarrow$ FALSE e1 y) e2 -- beta-step

=b> FALSE e1 e2 -- expand def TRUE

=d> ($\lambda x y \rightarrow y$) e1 e2 -- beta-step

=b> ($\lambda y \rightarrow y$) e2 -- beta-step

=b> e2

Boolean operators

- Now that we have ITE it's easy to define other Boolean operators:

```
let NOT = \b      -> ???
```

```
let AND = \b1 b2 -> ???
```

```
let OR  = \b1 b2 -> ???
```

Boolean operators

- Now that we have ITE it's easy to define other Boolean operators:

```
let NOT = \b      -> ITE b FALSE TRUE
```

```
let AND = \b1 b2 -> ITE b1 b2 FALSE
```

```
let OR  = \b1 b2 -> ITE b1 TRUE b2
```

Boolean operators

- Now that we have ITE it's easy to define other Boolean operators:

```
let NOT = \b      -> b FALSE TRUE
```

```
let AND = \b1 b2 -> b1 b2 FALSE
```

```
let OR  = \b1 b2 -> b1 TRUE b2
```

- (since ITE is redundant)
- *Which definition to do you prefer and why?*

Programming in λ -calculus

- Real languages have lots of features
 - **Booleans** [done]
 - Records (structs, tuples)
 - Numbers
 - **Functions** [we got those]
 - Recursion

λ -calculus: Records

- Let's start with records with two fields (aka pairs)?
- Well, what do we **do** with a pair?
 1. **Pack two** items into a pair, then
 2. **Get first** item, or
 3. **Get second** item.

Pairs: API

- We need to define three functions

```
let PAIR = \x y -> ???      -- Make a pair with x and y
                                -- { fst : x, snd : y }
let FST  = \p -> ???       -- Return first element
                                -- p.fst
let SND  = \p -> ???       -- Return second element
                                -- p.snd
```

such that

```
FST (PAIR apple banana) =~> apple
SND (PAIR apple banana) =~> banana
```

Pairs: Implementation

- A pair of x and y is just something that lets you pick between x and y ! (I.e. a function that takes a boolean and returns either x or y)

```
let PAIR = \x y -> (\b -> ITE b x y)
```

```
let FST  = \p -> p TRUE  -- call w/ TRUE, get 1st value
```

```
let SND  = \p -> p FALSE -- call w/ FALSE, get 2nd value
```


Exercise: Triples?

- How can we implement a record that contains **three** values?

```
let TRIPLE = \x y z -> ???
```

```
let FST3   = \t -> ???
```

```
let SND3   = \t -> ???
```

```
let TRD3   = \t -> ???
```

Exercise: Triples?

- How can we implement a record that contains **three** values?

```
let TRIPLE = \x y z -> PAIR x (PAIR y z)
```

```
let FST3   = \t -> FST t
```

```
let SND3   = \t -> FST (SND t)
```

```
let TRD3   = \t -> SND (SND t)
```

Programming in λ -calculus

- Real languages have lots of features
 - **Booleans** [done]
 - **Records (structs, tuples)** [done]
 - Numbers
 - **Functions** [we got those]
 - Recursion

λ -calculus: Numbers

- Let's start with **natural numbers** (0, 1, 2, ...)
- What do we do with natural numbers?
 1. **Count**: 0, inc
 2. **Arithmetic**: dec, +, -, *
 3. **Comparisons**: ==, <=, etc

Natural Numbers: API

- We need to define:
 - A family of numerals: ZERO, ONE, TWO, THREE, ...
 - Arithmetic functions: INC, DEC, ADD, SUB, MULT
 - Comparisons: IS_ZERO, EQ

Such that they respect all regular laws of arithmetic, e.g.

```
IS_ZERO ZERO           =~> TRUE
IS_ZERO (INC ZERO)    =~> FALSE
INC ONE               =~> TWO
...
```

Pairs: Implementation

- **Church numerals:** a *number* N is encoded as a combinator that *calls a function on an argument* N *times*

```
let ONE    = \f x -> f x
```

```
let TWO    = \f x -> f (f x)
```

```
let THREE  = \f x -> f (f (f x))
```

```
let FOUR   = \f x -> f (f (f (f x)))
```

```
let FIVE   = \f x -> f (f (f (f (f x))))
```

```
let SIX    = \f x -> f (f (f (f (f (f x))))))
```

```
...
```

QUIZ: Church Numerals

Which of these is a valid encoding of ZERO ? *

- A: $\text{let ZERO} = \lambda f x \rightarrow x$
- B: $\text{let ZERO} = \lambda f x \rightarrow f$
- C: $\text{let ZERO} = \lambda f x \rightarrow f x$
- D: $\text{let ZERO} = \lambda x \rightarrow x$
- E: None of the above



<http://tiny.cc/cse116-church-ind>

QUIZ: Church Numerals

Which of these is a valid encoding of ZERO ? *

- A: let ZERO = $\lambda f x \rightarrow x$
- B: let ZERO = $\lambda f x \rightarrow f$
- C: let ZERO = $\lambda f x \rightarrow f x$
- D: let ZERO = $\lambda x \rightarrow x$
- E: None of the above



<http://tiny.cc/cse116-church-grp>

λ -calculus: Increment

-- Call `f` on `x` one more time than `n` does

```
let INC = \n -> (\f x -> ???)
```

- Example

```
eval inc_zero :
```

```
INC ZERO
```

```
=d> (\n f x -> f (n f x)) ZERO
```

```
=b> \f x -> f (ZERO f x)
```

```
=*> \f x -> f x
```

```
=d> ONE
```

QUIZ: ADD

How shall we implement ADD? *

- A. let ADD = \n m -> n INC m
- B. let ADD = \n m -> INC n m
- C. let ADD = \n m -> n m INC
- D. let ADD = \n m -> n (m INC)
- E. let ADD = \n m -> n (INC m)



<http://tiny.cc/cse116-add-ind>

QUIZ: ADD

How shall we implement ADD? *

- A. let ADD = \n m -> n INC m
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<http://tiny.cc/cse116-add-grp>

λ -calculus: Addition

-- Call `f` on `x` exactly `n + m` times

```
let ADD = \n m -> n INC m
```

- Example

```
eval add_one_zero :
```

```
  ADD ONE ZERO
```

```
  ==> ONE
```

QUIZ: MULT

How shall we implement MULT? *

- A. let MULT = \n m -> n ADD m
- B. let MULT = \n m -> n (ADD m) ZERO
- C. let MULT = \n m -> m (ADD n) ZERO
- D. let MULT = \n m -> n (ADD m ZERO)
- E. let MULT = \n m -> (n ADD m) ZERO



<http://tiny.cc/cse116-mult-ind>

QUIZ: MULT

How shall we implement MULT? *

- A. let MULT = \n m -> n ADD m
- B. let MULT = \n m -> n (ADD m) ZERO
- C. let MULT = \n m -> m (ADD n) ZERO
- D. let MULT = \n m -> n (ADD m ZERO)
- E. let MULT = \n m -> (n ADD m) ZERO



<http://tiny.cc/cse116-mult-grp>

λ -calculus: Multiplication

-- Call `f` on `x` exactly `n * m` times

```
let MULT = \n m -> n (ADD m) ZERO
```

- Example

```
eval two_times_one :
```

```
MULT TWO ONE
```

```
=~> TWO
```

Programming in λ -calculus

- Real languages have lots of features
 - **Booleans** [done]
 - **Records (structs, tuples)** [done]
 - **Numbers** [done]
 - **Functions** [we got those]
 - **Recursion**

λ -calculus: Recursion

- I want to write a function that sums up natural numbers up to n :

$\backslash n \rightarrow \dots$

-- $1 + 2 + \dots + n$

QUIZ: SUM

Is this a correct implementation of SUM? *

```
let SUM = \n -> ITE (ISZ n)  
                ZERO  
                (ADD n (SUM (DEC n)))
```

A. Yes

B. No



<http://tiny.cc/cse116-sum-ind>

QUIZ: SUM

Is this a correct implementation of SUM? *

```
let SUM = \n -> ITE (ISZ n)  
                ZERO  
                (ADD n (SUM (DEC n)))
```

A. Yes

B. No



<http://tiny.cc/cse116-sum-grp>

λ -calculus: Recursion

- No! Named terms in Elsa are just syntactic sugar
- To translate an Elsa term to λ -calculus: replace each name with its definition

```
\n -> ITE (ISZ n)
        ZERO
        (ADD n (SUM (DEC n))) -- But SUM is
                               -- not a thing!
```

- **Recursion:** Inside this function I want to call the same function on **DEC n**
- Looks like we can't do recursion, because it requires being able to refer to functions *by name*, but in λ -calculus functions are *anonymous*.
- ***Right?***

λ -calculus: Recursion

- Think again!
- Recursion: ~~Inside this function I want to call the same function on DEC n~~
 - Inside this function I want to call a function on DEC n
 - And BTW, I want it to be the same function
- Step 1: Pass in the function to call “recursively”

```
let STEP =  
  \rec ->  
    \n -> ITE (ISZ n)  
              ZERO  
              (ADD n (rec (DEC n))) -- Call some rec
```

λ -calculus: Recursion

- Step 1: Pass in the function to call “recursively”

```
let STEP =  
  \rec ->  
    \n -> ITE (ISZ n)  
              ZERO  
              (ADD n (rec (DEC n))) -- Call some rec
```

- Step 2: Do something clever to `STEP`, so that the function passed as `rec` itself becomes

```
\n -> ITE (ISZ n) ZERO (ADD n (rec (DEC n)))
```

λ -calculus: Fixpoint Combinator

- **Wanted:** a combinator **FIX** such that **FIX STEP** calls **STEP** with itself as the first argument:

```
FIX STEP
=*> STEP (FIX STEP)
```

(In math: a *fixpoint* of a function $f(x)$ is a point x , such that $f(x) = x$)

- Once we have it, we can define:

```
let SUM = FIX STEP
```

- Then by property of **FIX** we have:

```
SUM =*> STEP SUM -- (1)
```

λ -calculus: Fixpoint Combinator

eval sum_one:

SUM ONE

=*> STEP SUM ONE -- (1)

=d> (\rec n -> ITE (ISZ n) ZERO (ADD n (rec (DEC n)))) SUM ONE

=b> (\n -> ITE (ISZ n) ZERO (ADD n (SUM (DEC n)))) ONE

-- ^^ the magic happened!

=b> ITE (ISZ ONE) ZERO (ADD ONE (SUM (DEC ONE)))

=*> ADD ONE (SUM ZERO) -- def of ISZ, ITE, DEC, ...

=*> ADD ONE (STEP SUM ZERO) -- (1)

=d> ADD ONE

((\rec n -> ITE (ISZ n) ZERO (ADD n (rec (DEC n)))) SUM ZERO)

=b> ADD ONE ((\n -> ITE (ISZ n) ZERO (ADD n (SUM (DEC n)))) ZERO)

=b> ADD ONE (ITE (ISZ ZERO) ZERO (ADD ZERO (SUM (DEC ZERO))))

=b> ADD ONE ZERO

=~> ONE

λ -calculus: Fixpoint Combinator

- So how do we define **FIX**?
- Remember Ω ? It *replicates itself!*

$$\begin{aligned} & (\lambda x \rightarrow x x) (\lambda x \rightarrow x x) \\ =b> & (\lambda x \rightarrow x x) (\lambda x \rightarrow x x) \end{aligned}$$

- We need something similar but more involved.

λ -calculus: Fixpoint Combinator

- The Y combinator discovered by Haskell Curry:

```
let FIX = \stp -> (\x -> stp (x x)) (\x -> stp (x x))
```

- How does it work?

```
eval fix_step:
```

```
FIX STEP
```

```
=d> (\stp -> (\x -> stp (x x)) (\x -> stp (x x))) STEP
```

```
=b> (\x -> STEP (x x)) (\x -> STEP (x x))
```

```
=b> STEP ((\x -> STEP (x x)) (\x -> STEP (x x)))
```

```
--          ^^^^^^^^^^^^ this is FIX STEP ^^^^^^^^^^^^^^^
```

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Next time: Intro to Haskell

