CSE 114A: Fall 2023

Foundations of Programming Languages

Environments and closures

Owen Arden

UC Santa Cruz

Roadmap

Past weeks:

- How do we *use* a functional language?

Next weeks:

- How do we *implement* a functional language?
- … in a functional language (of course)

WHY??

- *Master* the concepts of functional languages by implementing them!
- *Practice* problem solving using Haskell

This week: Interpreter

- How do we *evaluate* a program given its abstract syntax tree (AST)?
- How do we *prove properties* about our interpreter (e.g. that certain programs never crash)?

The Nano Language

Features of Nano:

1. Arithmetic expressions
2. Variables and let-bindings
3. Functions
4. Recursion

Reminder: Calculator

Arithmetic expressions:

```
e ::= n
    | e1 + e2
    | e1 - e2
    | e1 * e2
```

Example:

```
4 + 13
==> 17
```

Reminder: Calculator

Haskell datatype to *represent* arithmetic expressions:

```haskell
data Expr = Num Int
          | Add Expr Expr
          | Sub Expr Expr
          | Mul Expr Expr
```

Haskell function to *evaluate* an expression:

```haskell
eval :: Expr -> Int
eval (Num n)     = n
eval (Add e1 e2) = eval e1 + eval e2
eval (Sub e1 e2) = eval e1 - eval e2
eval (Mul e1 e2) = eval e1 * eval e2
```

Reminder: Calculator

Alternative representation:

```haskell
data Binop = Add | Sub | Mul

data Expr = Num Int              -- number
          | Bin Binop Expr Expr  -- binary expression
```

Evaluator for alternative representation:

```haskell
eval :: Expr -> Int
eval (Num n)         = n
eval (Bin Add e1 e2) = eval e1 + eval e2
eval (Bin Sub e1 e2) = eval e1 - eval e2
eval (Bin Mul e1 e2) = eval e1 * eval e2
```

The Nano Language

Features of Nano:

1. Arithmetic expressions **[done]**
2. Variables and let-bindings
3. Functions
4. Recursion

Extension: variables

Let's add variables and **let** bindings!

```
e ::= n | x
    | e1 + e2 | e1 - e2 | e1 * e2
    | let x = e1 in e2
```

Example:

```
let x = 4 + 13 in   -- 17
let y = 7 - 5 in    -- 2
x * y

==> 34
```

Extension: variables

Haskell representation:

```haskell
data Expr = Num Int            -- number
          | ???                -- variable
          | Bin Binop Expr Expr -- binary expression
          | ???                -- let expression
```

Extension: variables

```haskell
type Id = String

data Expr = Num Int              -- number
          | Var Id               -- variable
          | Bin Binop Expr Expr  -- binary expression
          | Let Id Expr Expr     -- let expression
```

Haskell function to *evaluate* an expression:

```haskell
eval :: Expr -> Int
eval (Num n)        = n
eval (Var x)        = ???
...
```

Extension: variables

```haskell
type Id = String

data Expr = Num Int              -- number
```

How do we evaluate a variable?

We have to remember
which *value* it was bound to!

```haskell
eval :: Exp
eval (Num n
eval (Var x)           = ???
...
```

Environment

An expression is evaluated in an **environment**, which maps all its *free variables* to *values*

Examples:

x * y

=[x:17, y:2]=> 34

```
┌─────────────────────────────────────┐
│ • How should we represent the       │
│   environment?                      │
│ • Which operations does it support? │
└─────────────────────────────────────┘
```

x * y

=[x:17]=> Error: unbound variable y

x * (**let** y = 2 **in** y)

=[x:17]=> 34

Extension: variables

What does this evaluate to? *

```
let x = 5 in
let y = x + z in
let z = 10 in
y
```

○ (A) 15

○ (B) 5

○ (C) Error: unbound variable x

○ (D) Error: unbound variable y

○ (E) Error: unbound variable z

http://tiny.cc/cse116-vars-ind

Extension: variables

What does this evaluate to? *

```
let x = 5 in
let y = x + z in
let z = 10 in
y
```

○ (A) 15

○ (B) 5

○ (C) Error: unbound variable x

○ (D) Error: unbound variable y

○ (E) Error: unbound variable z

http://tiny.cc/cse116-vars-grp

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Environment: API

To evaluate **let** x = e1 **in** e2 in env:

- evaluate e2 in an **extended** environment env + [x:v]
- where v is the result of evaluating e1

To evaluate x in env:

- **lookup** the most recently added binding for x

```
type Value = Int

data Env = ...  -- representation not that important

-- | Add a new binding
add    :: Id -> Value -> Env -> Env

-- | Lookup the most recently added binding
lookup :: Id -> Env -> Value
```

Evaluating expressions

Back to our expressions... now with environments!

```haskell
data Expr = Num Int             -- number
          | Var Id              -- variable
          | Bin Binop Expr Expr -- binary expression
          | Let Id Expr Expr    -- let expression
```

Evaluating expressions

Haskell function to *evaluate* an expression:

```haskell
eval :: Env -> Expr -> Value
eval env (Num n)        = n
eval env (Var x)        = lookup x env
eval env (Bin op e1 e2) = f v1 v2
  where
    v1 = eval env e1
    v2 = eval env e2
    f = case op of
          Add -> (+)
          Sub -> (-)
          Mul -> (*)
eval env (Let x e1 e2)  = eval env' e2
  where
    v    = eval env e1
    env' = add x v env
```

Example evaluation

Nano expression

```
let x = 1 in
let y = (let x = 2 in x) + x in
let x = 3 in
x + y
```

is represented in Haskell as:

```
exp1 =    Let "x"
            (Num 1)                              exp2
            (Let "y"
              (Add                               exp3
                (Let "x" (Num 2) (Var x))
          exp4 (Var x))
              (Let "x"                           exp5
                (Num 3)
                (Add (Var x) (Var y)))))
```

Example evaluation

```
eval [] exp1
=> eval []                             (Let "x" (Num 1) exp2)
=> eval [("x",eval [] (Num 1))] exp2
=> eval [("x",1)]
    (Let "y" (Add exp3 exp4) exp5)
=> eval [("y",(eval [("x",1)] (Add exp3 exp4))), ("x",1)]
    exp5
=> eval [("y",(eval [("x",1)] (Let "x" (Num 2) (Var "x"))
            + eval [("x",1)] (Var "x"))), ("x",1)]
    exp5
=> eval [("y",(eval [("x",2), ("x",1)] (Var "x") -- new binding for x
            + 1)), ("x",1)]
    exp5
=> eval [("y",(2  -- use latest binding for x
            + 1)), ("x",1)]
    exp5
=> eval [("y",3), ("x",1)]
    (Let "x" (Num 3) (Add (Var "x") (Var "y")))
```

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Example evaluation

```
=> eval [("y",3), ("x",1)]
     (Let "x" (Num 3) (Add (Var "x") (Var "y")))
=> eval [("x",3), ("y",3), ("x",1)]        -- new binding for x
     (Add (Var "x") (Var "y"))
=> eval [("x",3), ("y",3), ("x",1)] (Var "x")
 + eval [("x",3), ("y",3), ("x",1)] (Var "y")
=> 3 + 3
=> 6
```

Example evaluation

Same evaluation in a simplified format (Haskell `Expr` terms replaced by their "pretty-printed version"):

```
    eval []
    {let x = 1 in let y = (let x = 2 in x) + x in let x = 3 in x + y}
=> eval [x:(eval [] 1)]
               {let y = (let x = 2 in x) + x in let x = 3 in x + y}
=> eval [x:1]
               {let y = (let x = 2 in x) + x in let x = 3 in x + y}
=> eval [y:(eval [x:1] {(let x = 2 in x) + x}), x:1]
                                        {let x = 3 in x + y}
=> eval [y:(eval [x:1]  {let x = 2 in x} + eval [x:1] {x}), x:1]
                                        {let x = 3 in x + y}

        -- new binding for x:
=> eval [y:(eval [x:2,x:1] {x}          + eval [x:1] {x}), x:1]
                                        {let x = 3 in x + y}

    -- use latest binding for x:
=> eval [y:(                 2          + eval [x:1] {x}), x:1]
                                        {let x = 3 in x + y}

=> eval [y:(                 2          + 1)            , x:1]
                                        {let x = 3 in x + y}
```

Example evaluation

```
=> eval [y:(              2          + 1)          , x:1]
                                        {let x = 3 in x + y}

=> eval [y:3, x:1]
                                        {let x = 3 in x + y}
   -- new binding for x:
=> eval [x:3, y:3, x:1]
                                                      {x + y}

=> eval [x:3, y:3, x:1] x + eval [x:3, y:3, x:1] y
   -- use latest binding for x:
=> 3 + 3
=> 6
```

Runtime errors

Haskell function to *evaluate* an expression:

```
eval :: Env -> Expr -> Value
eval env (Num n)        = n
eval env (Var x)        = lookup x env -- can fail!
eval env (Bin op e1 e2) = f v1 v2
  where
    v1 = eval env e1
    v2 = eval env e2
    f = case op of
          Add -> (+)
          Sub -> (-)
          Mul -> (*)
eval env (Let x e1 e2)  = eval env' e2
  where
    v    = eval env e1
    env' = add x v env
```

How do we make sure `lookup` doesn't cause a run-time error?

Free vs bound variables

In `eval env e`, `env` must contain bindings for *all free variables* of `e`!

- an occurrence of `x` is **free** if it is not **bound**
- an occurrence of `x` is **bound** if it's inside `e2` where **let** `x` `=` `e1` **in** `e2`
- evaluation succeeds when an expression is **closed**!

QUIZ

Which variables are free in the expression? *

```
let y = (let x = 2 in x) + x in
let x = 3 in
x + y
```

○ (A) None

○ (B) x

○ (C) y

○ (D) x y

http://tiny.cc/cse116-free-ind

QUIZ

Which variables are free in the expression? *

```
let y = (let x = 2 in x) + x in
let x = 3 in
x + y
```

○ (A) None

○ (B) x

○ (C) y

○ (D) x y

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The Nano Language

Features of Nano:

1. Arithmetic expressions **[done]**
2. Variables and let-bindings **[done]**
3. Functions
4. Recursion

Extension: functions

Let's add lambda abstraction and function application!

```
e ::= n | x
    | e1 + e2 | e1 - e2 | e1 * e2
    | let x = e1 in e2
    | \x -> e    -- abstraction
    | e1 e2      -- application
```

Example:

```
let c = 42 in
let cTimes = \x -> c * x in
cTimes 2
==> 84
```

Extension: functions

Haskell representation:

```haskell
data Expr = Num Int              -- number
          | Var Id               -- variable
          | Bin Binop Expr Expr  -- binary expression
          | Let Id Expr Expr     -- let expression
          | ???                  -- abstraction
          | ???                  -- application
```

Extension: functions

Haskell representation:

```haskell
data Expr = Num Int              -- number
          | Var Id               -- variable
          | Bin Binop Expr Expr  -- binary expression
          | Let Id Expr Expr     -- let expression
          | Lam Id Expr          -- abstraction
          | App Expr Expr        -- application
```

Extension: functions

Example:

```
let c = 42 in
let cTimes = \x -> c * x in
cTimes 2
```

represented as:

```
Let "c"
  (Num 42)
  (Let "cTimes"
    (Lam "x" (Mul (Var "c") (Var "x")))
    (App (Var "cTimes") (Num 2)))
```

Extension: functions

Example:

```
let c = 42 in
let cTimes = \x -> c * x in
cTimes 2
```

How should we evaluate this expression?

```
    eval []
     {let c = 42 in let cTimes = \x -> c * x in cTimes 2}
=> eval [c:42]
                    {let cTimes = \x -> c * x in cTimes 2}
=> eval [cTimes:???, c:42]
                                            {cTimes 2}
```

What is the **value** of cTimes???

Rethinking our values

Until now: a program *evaluates* to an integer (or fails)

```
type Value = Int

type Env = [(Id, Value)]

eval :: Env -> Expr -> Value
```

Rethinking our values

What do these programs evaluate to?

```
(1)
\x -> 2 * x
==> ???
```

```
(2)
let f = \x -> \y -> 2 * (x + y) in
f 5
==> ???
```

Conceptually, (1) evaluates to itself (not exactly, see later). while (2) evaluates to something equivalent to `\y -> 2 * (5 + y)`

Rethinking our values

Now: a program evaluates to an integer or *a lambda abstraction* (or fails)

- Remember: functions are *first-class* values

Let's change our definition of values!

```haskell
data Value = VNum Int
           | VLam ??? -- What info do we need to store?

-- Other types stay the same
type Env = [(Id, Value)]

eval :: Env -> Expr -> Value
```

Function values

How should we represent a function value?

```
let c = 42 in
let cTimes = \x -> c * x in
cTimes 2
```

We need to store enough information about `cTimes` so that we can later evaluate any *application* of `cTimes` (like `cTimes 2`)!

First attempt:

```
data Value = VNum Int
           | VLam Id Expr -- formal + body
```

Function values

Let's try this!

```
eval []
  {let c = 42 in let cTimes = \x -> c * x in cTimes 2}
=> eval [c:42]
                    {let cTimes = \x -> c * x in cTimes 2}
=> eval [cTimes:(\x -> c*x), c:42]
                                        {cTimes 2}
  -- evaluate the function:
=> eval [cTimes:(\x -> c*x), c:42]
                            {(\x -> c * x) 2}
  -- evaluate the argument, bind to x, evaluate body:
=> eval [x:2, cTimes:(\x -> c*x), c:42]
                                {c * x}
=>                              42 * 2
=>                              84
```

Looks good… can you spot a problem?

QUIZ

What should this evaluate to? *

```
let c = 42 in
let cTimes = \x -> c * x in -- but which c???
let c = 5 in
cTimes 2
```

○ (A) 84

○ (B) 10

○ (C) Error: multiple definitions of c

http://tiny.cc/cse116-cscope-ind

QUIZ

What should this evaluate to? *

```
let c = 42 in
let cTimes = \x -> c * x in -- but which c???
let c = 5 in
cTimes 2
```

○ (A) 84

○ (B) 10

○ (C) Error: multiple definitions of c

http://tiny.cc/cse116-cscope-grp

Static vs Dynamic Scoping

What we want:

```
let c = 42 in
let cTimes = \x -> c * x in
let c = 5 in
cTimes 2
=> 84
```

Lexical (or **static**) scoping:

- each occurrence of a variable refers to the most recent binding *in the program text*
- definition of each variable is unique and known *statically*
- good for readability and debugging: don't have to figure out where a variable got "assigned"

Static vs Dynamic Scoping

What we **don't** want:

```
let c = 42 in
let cTimes = \x -> c * x in
let c = 5 in
cTimes 2
=> 10
```

Dynamic scoping:

- each occurrence of a variable refers to the most recent binding *during program execution*
- can't tell where a variable is defined just by looking at the function body
- nightmare for readability and debugging:

Static vs Dynamic Scoping

Dynamic scoping:

- each occurrence of a variable refers to the most recent binding *during program execution*
- can't tell where a variable is defined just by looking at the function body
- nightmare for readability and debugging:

```
let cTimes = \x -> c * x in
let c = 5 in
let res1 = cTimes 2 in -- ==> 10
let c = 10 in
let res2 = cTimes 2 in -- ==> 20!!!
res2 - res1
```

Function values

```haskell
data Value = VNum Int
           | VLam Id Expr -- formal + body
```

This representation can only implement dynamic scoping!

```haskell
let c = 42 in

let cTimes = \x -> c * x in

let c = 5 in

cTimes 2
```

evaluates as:

```haskell
eval []
{let c = 42 in let cTimes = \x -> c * x in let c = 5 in cTimes 2}
```

Function values

```
eval []
{let c = 42 in let cTimes = \x -> c * x in let c = 5 in cTimes 2}
=> eval [c:42]
                    {let cTimes = \x -> c * x in let c = 5 in cTimes 2}
=> eval [cTimes:(\x -> c*x), c:42]
                                        {let c = 5 in cTimes 2}
=> eval [c:5, cTimes:(\x -> c*x), c:42]
                                            {cTimes 2}
=> eval [c:5, cTimes:(\x -> c*x), c:42]
                                        {(\x -> c * x) 2}
=> eval [x:2, c:5, cTimes:(\x -> c*x), c:42]
                                            {c * x}
  -- latest binding for c is 5!
=>                                           5 * 2
=>                                           10
```

Lesson learned: need to remember what `c` was bound to when `cTimes` was defined!

- i.e. "freeze" the environment at function definition

Closures

To implement lexical scoping, we will represent function values as *closures*

Closure = *lambda abstraction* (formal + body) + *environment* at function definition

```haskell
data Value = VNum Int
           | VClos Env Id Expr -- env + formal + body
```

Closures

Our example:

```
    eval []
    {let c = 42 in let cTimes = \x -> c * x in let c = 5 in cTimes 2}
=> eval [c:42]
                    {let cTimes = \x -> c * x in let c = 5 in cTimes 2}
    -- remember current env:
=> eval [cTimes:<[c:42], \x -> c*x>, c:42]
                                        {let c = 5 in cTimes 2}
=> eval [c:5, cTimes:<[c:42], \x -> c*x>, c:42]
                                        {cTimes 2}
=> eval [c:5, cTimes:<[c:42], \x -> c*x>, c:42]
                                        {<[c:42], \x -> c * x> 2}
    -- restore env to the one inside the closure, then bind 2 to x:
=> eval [x:2, c:42]
                                                {c * x}
=>                                              42 * 2
=>                                              84
```

QUIZ

Which variables should be saved in the closure environment of f? *

```
let a = 20 in
let f =
   \x -> let y = x + 1 in
         let g = \z -> y + z in
         a + g x
   in ...
```

○ (A) a

○ (B) a x

○ (C) y g

○ (D) a y g

○ (E) a x y g z

http://tiny.cc/cse116-env-ind

QUIZ

Which variables should be saved in the closure environment of f? *

```
let a = 20 in
let f =
  \x -> let y = x + 1 in
        let g = \z -> y + z in
        a + g x
  in ...
```

○ (A) a

○ (B) a x

○ (C) y g

○ (D) a y g

○ (E) a x y g z

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Free vs bound variables

- An occurrence of x is **free** if it is not **bound**
- An occurrence of x is **bound** if it's inside
  - e2 where **let** x = e1 **in** e2
  - e where \x -> e
- A closure environment has to save *all free variables* of a function definition!

```
let a = 20 in
let f =
  \x -> let y = x + 1 in
        let g = \z -> y + z in
        a + g x -- a is the only free variable!
  in ...
```

Evaluator

Let's modify our evaluator to handle functions!

```haskell
data Value = VNum Int
           | VClos Env Id Expr -- env + formal + body

eval :: Env -> Expr -> Value
eval env (Num n)        = VNum n -- must wrap in VNum now!
eval env (Var x)        = lookup x env
eval env (Bin op e1 e2) = VNum (f v1 v2)
  where
    (VNum v1) = eval env e1
    (VNum v2) = eval env e2
    f = ... -- as before
eval env (Let x e1 e2) = eval env' e2
  where
    v = eval env e1
    env' = add x v env
eval env (Lam x body) = ??? -- construct a closure
eval env (App fun arg) = ??? -- eval fun, then arg, then apply
```

Evaluator

Evaluating functions:

- **Construct a closure**: save environment at function definition
- **Apply a closure**: restore saved environment, add formal, evaluate the body

```haskell
eval :: Env -> Expr -> Value
...
eval env (Lam x body) = VClos env x body
eval env (App fun arg) = eval bodyEnv body
  where
    (VClos closEnv x body) = eval env fun -- eval function to closure
    vArg                   = eval env arg -- eval argument
    bodyEnv                = add x vArg closEnv
```

Evaluator

Evaluating functions:

- **Construct a closure**: save environment at function definition
- **Apply a closure**: restore saved environment, add formal, evaluate the body

```
eval :: Env -> Expr -> Value
...
eval env (Lam x body) = VClos env x body
eval env (App fun arg) =
    let vArg = eval env arg in -- eval argument
    let (VClos closEnv x body) = (eval env fun) in
    let bodyEnv = add x vArg closEnv in
    eval bodyEnv body
```

Quiz

With eval as defined above, what does this evaluate to? *

```
let f = \x -> x + y in
let y = 10 in
f 5
```

○ (A) 15

○ (B) 5

○ (C) Error: unbound variable x

○ (D) Error: unbound variable y

○ (E) Error: unbound variable f

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Quiz

With eval as defined above, what does this evaluate to? *

```
let f = \x -> x + y in
let y = 10 in
f 5
```

○ (A) 15

○ (B) 5

○ (C) Error: unbound variable x

○ (D) Error: unbound variable y

○ (E) Error: unbound variable f

http://tiny.cc/cse116-enveval-grp

Evaluator

```
  eval []
    {let f = \x -> x + y in let y = 10 in f 5}
=> eval [f:<[], \x -> x + y>]
                            {let y = 10 in f 5}
=> eval [y:10, f:<[], \x -> x + y>]
                                      {f 5}
=> eval [y:10, f:<[], \x -> x + y>]
                      {<[], \x -> x + y> 5}
=> eval [x:5] -- env got replaced by closure env + formal!
                              {x + y}  -- y is unbound!
```

Quiz

With eval as defined above, what does this evaluate to? *

```
let f = \n -> n * f (n - 1) in
f 5
```

○ (A) 120

○ (B) Evaluation does not terminate

○ (C) Error: unbound variable f

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Quiz

With eval as defined above, what does this evaluate to? *

```
let f = \n -> n * f (n - 1) in
f 5
```

○ (A) 120

○ (B) Evaluation does not terminate

○ (C) Error: unbound variable f

http://tiny.cc/cse116-enveval2-grp

Evaluator

```
   eval []
        {let f = \n -> n * f (n - 1) in f 5}
=> eval [f:<[], \n -> n * f (n - 1)>]
                                    {f 5}
=> eval [f:<[], \n -> n * f (n - 1)>]
              {<[], \n -> n * f (n - 1)> 5}
=> eval [n:5] -- env got replaced by closure env + formal!
                    {n * f (n - 1)} -- f is unbound!
```

Lesson learned: to support recursion, we need a different way of constructing the closure environment!