

CSE 114A: Fall 2021

Foundations of Programming Languages

Formalizing Nano

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Based on course materials developed by Nadia Polikarpova

Formalizing Nano

Goal: we want to guarantee properties about programs, such as:

- evaluation is deterministic
- all programs terminate
- certain programs never fail at run time
- etc.

To prove theorems about programs we first need to define formally

- their *syntax* (what programs look like)
- their *semantics* (what it means to run a program)

Let's start with Nano1 (Nano w/o functions) and prove some stuff!

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Nano1: Syntax

We need to define the syntax for *expressions (terms)* and *values* using a grammar:

```
e ::= x | v           -- expressions
    | e1 + e2
    | let x = e1 in e2
```

```
v ::= n           -- values
```

where $n \in \mathbb{N}$, $x \in \text{Var}$

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Nano1: Operational Semantics

Operational semantics defines how to execute a program step by step

Let's define a *step relation* (reduction relation) $e \Rightarrow e'$

- “expression e makes a step (reduces in one step) to an expression e' ”

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Nano1: Operational Semantics

We define the step relation *inductively* through a set of *rules*:

[Add-L]
$$\frac{e1 \Rightarrow e1' \quad \text{-- premise}}{e1 + e2 \Rightarrow e1' + e2 \quad \text{-- conclusion}}$$

[Add-R]
$$\frac{e2 \Rightarrow e2'}{n1 + e2 \Rightarrow n1 + e2'}$$

[Add] $n1 + n2 \Rightarrow n \quad \text{where } n == n1 + n2$

[Let-Def]
$$\frac{e1 \Rightarrow e1'}{\text{let } x = e1 \text{ in } e2 \Rightarrow \text{let } x = e1' \text{ in } e2}$$

[Let] $\text{let } x = v \text{ in } e2 \Rightarrow e2[x := v]$

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Nano1: Operational Semantics

Here $e[x := v]$ is a value substitution:

$x[x := v] = v$
 $y[x := v] = y \quad \text{-- assuming } x \neq y$
 $n[x := v] = n$
 $(e1 + e2)[x := v] = e1[x := v] + e2[x := v]$
 $(\text{let } x = e1 \text{ in } e2)[x := v] = \text{let } x = e1[x := v] \text{ in } e2$
 $(\text{let } y = e1 \text{ in } e2)[x := v] = \text{let } y = e1[x := v] \text{ in } e2[x := v]$

Do not have to worry about capture, because v is a value (has no free variables!)

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Nano1: Operational Semantics

A reduction is *valid* if we can build its **derivation** by “stacking” the rules:

```
[Add] -----  
      1 + 2 => 3  
[Add-L] -----  
      (1 + 2) + 5 => 3 + 5
```

Do we have rules for all kinds of expressions?

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Nano1: Operational Semantics

We define the step relation *inductively* through a set of *rules*:

```
[Add-L] 
$$\frac{e1 \Rightarrow e1'}{e1 + e2 \Rightarrow e1' + e2} \quad \begin{array}{l} \text{-- premise} \\ \text{-- conclusion} \end{array}$$
  
[Add-R] 
$$\frac{e2 \Rightarrow e2'}{n1 + e2 \Rightarrow n1 + e2'}$$
  
[Add] 
$$n1 + n2 \Rightarrow n \quad \text{where } n == n1 + n2$$
  
[Let-Def] 
$$\frac{e1 \Rightarrow e1'}{\text{let } x = e1 \text{ in } e2 \Rightarrow \text{let } x = e1' \text{ in } e2}$$
  
[Let] 
$$\text{let } x = v \text{ in } e2 \Rightarrow e2[x := v]$$

```

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1. Normal forms

There are no reduction rules for:

- n
- x

Both of these expressions are *normal forms* (cannot be further reduced), however:

- n is a *value*
 - intuitively, corresponds to successful evaluation
- x is *not* a value
 - intuitively, corresponds to a run-time error!
 - we say the program x is *stuck*

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2. Evaluation order

In $e1 + e2$, which side should we evaluate first?

In other words, which one of these reductions is valid (or both)?

1. $(1 + 2) + (4 + 5) \Rightarrow 3 + (4 + 5)$
2. $(1 + 2) + (4 + 5) \Rightarrow (1 + 2) + 9$

Reduction (1) is *valid* because we can build a *derivation* using the rules:

```
[Add] -----
      1 + 2 => 3
[Add-L] -----
(1 + 2) + (4 + 5) => 3 + (4 + 5)
```

Reduction (2) is *invalid* because we cannot build a derivation:

- there is *no rule* whose conclusion matches this reduction!

```
      ???
[???] -----
(1 + 2) + (4 + 5) => (1 + 2) + 9
```

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QUIZ

If these are the only rules for let bindings, which reductions are valid? *

```
      e1 => e1'
[Let-Def] -----
let x = e1 in e2 => let x = e1' in e2
```

```
[Let] let x = v in e2 => e2[x := v]
```

- (A) $(\text{let } x = 1 + 2 \text{ in } 4 + 5 + x) \Rightarrow (\text{let } x = 3 \text{ in } 4 + 5 + x)$
- (B) $(\text{let } x = 1 + 2 \text{ in } 4 + 5 + x) \Rightarrow (\text{let } x = 1 + 2 \text{ in } 9 + x)$
- (C) $(\text{let } x = 1 + 2 \text{ in } 4 + 5 + x) \Rightarrow (4 + 5 + 1 + 2)$
- (D) A and B
- (E) All of the above



<http://tiny.cc/cse116-reduce-ind>

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QUIZ

If these are the only rules for let bindings, which reductions are valid? *

```
      e1 => e1'
[Let-Def] -----
let x = e1 in e2 => let x = e1' in e2
```

```
[Let] let x = v in e2 => e2[x := v]
```

- (A) $(\text{let } x = 1 + 2 \text{ in } 4 + 5 + x) \Rightarrow (\text{let } x = 3 \text{ in } 4 + 5 + x)$
- (B) $(\text{let } x = 1 + 2 \text{ in } 4 + 5 + x) \Rightarrow (\text{let } x = 1 + 2 \text{ in } 9 + x)$
- (C) $(\text{let } x = 1 + 2 \text{ in } 4 + 5 + x) \Rightarrow (4 + 5 + 1 + 2)$
- (D) A and B
- (E) All of the above



<http://tiny.cc/cse116-reduce-grp>

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Evaluation relation

Like in λ -calculus, we define the multi-step reduction relation $e \Rightarrow^* e'$:

$e \Rightarrow^* e'$ iff there exists a sequence of expressions e_1, \dots, e_n such that

- $e = e_1$
- $e_n = e'$
- $e_i \Rightarrow e_{i+1}$ for each i in $[0..n)$

Example:

$(1 + 2) + (4 + 5)$

$\Rightarrow^* 3 + 9$

because

$(1 + 2) + (4 + 5)$

$\Rightarrow 3 + (4 + 5)$

$\Rightarrow 3 + 9$

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Evaluation relation

Now we define the evaluation relation $e \rightsquigarrow e'$:

$e \rightsquigarrow e'$ iff

- $e \Rightarrow^* e'$
- e' is in normal form

Example:

$(1 + 2) + (4 + 5)$

$\rightsquigarrow 12$

because

$(1 + 2) + (4 + 5)$

$\Rightarrow 3 + (4 + 5)$

$\Rightarrow 3 + 9$

$\Rightarrow 12$

and 12 is a *value* (normal form)

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Theorems about Nano1

Let's prove something about Nano1!

1. Every Nano1 program terminates
2. Closed Nano1 programs don't get stuck
3. *Corollary (1 + 2)*: Every closed Nano1 program evaluates to a value

How do we prove theorems about languages?

By induction.

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Mathematical induction in PL

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1. Induction on natural numbers

To prove $\forall n.P(n)$ we need to prove:

- *Base case*: $P(0)$
- *Inductive case*: $P(n + 1)$ assuming the *induction hypothesis* (IH): that $P(n)$ holds

Compare with inductive definition for natural numbers:

```
data Nat = Zero    -- base case
         | Succ Nat -- inductive case
```

No reason why this would only work for natural numbers...

In fact we can do induction on *any* inductively defined mathematical object (= any datatype)!

- lists
- trees
- programs (terms)
- etc

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2. Induction on terms

```
e ::= n | x
    | e1 + e2
    | let x = e1 in e2
```

To prove $\forall e.P(e)$ we need to prove:

- *Base case 1*: $P(n)$
- *Base case 2*: $P(x)$
- *Inductive case 1*: $P(e1 + e2)$ assuming the IH: that $P(e1)$ and $P(e2)$ hold
- *Inductive case 2*: $P(\text{let } x = e1 \text{ in } e2)$ assuming the IH: that $P(e1)$ and $P(e2)$ hold

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3. Induction on derivations

Our reduction relation \Rightarrow is also defined *inductively!*

- Axioms are base cases
- Rules with premises are inductive cases

To prove $\forall e, e'. P(e \Rightarrow e')$ we need to prove:

- *Base cases:* [Add], [Let]
- *Inductive cases:* [Add-L], [Add-R], [Let-Def] assuming the IH: that P holds of their premise

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Theorem: Termination

Theorem I [Termination]: For any expression e there exists e' such that $e \Rightarrow^* e'$.

Proof idea: let's define the *size* of an expression such that

- size of each expression is positive
- each reduction step strictly decreases the size

Then the length of the execution sequence for e is *bounded* by the size of $e!$

size n = ???
size x = ???
size $(e1 + e2)$ = ???
size $(\text{let } x = e1 \text{ in } e2)$ = ???

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Theorem: Termination

Term size:

size n = 1
size x = 1
size $(e1 + e2)$ = size $e1$ + size $e2$
size $(\text{let } x = e1 \text{ in } e2)$ = size $e1$ + size $e2$

Lemma 1: For any e , size $e > 0$.

Proof: By induction on the *term* e .

- *Base case 1:* size $n = 1 > 0$
- *Base case 2:* size $x = 1 > 0$
- *Inductive case 1:* size $(e1 + e2) = \text{size } e1 + \text{size } e2 > 0$ because size $e1 > 0$ and size $e2 > 0$ by IH.
- *Inductive case 2:* similar.

QED.

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Theorem: Termination

Lemma 2: For any e, e' such that $e \Rightarrow e'$, $\text{size } e' < \text{size } e$.

Proof: By induction on the *derivation* of $e \Rightarrow e'$.

Base case [Add].

- Given: the root of the derivation is
[Add]: $n1 + n2 \Rightarrow n$ where $n = n1 + n2$
- To prove: $\text{size } n < \text{size } (n1 + n2)$
- $\text{size } n = 1 + 2 = \text{size } (n1 + n2)$

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Theorem: Termination

Lemma 2: For any e, e' such that $e \Rightarrow e'$, $\text{size } e' < \text{size } e$.

Inductive case [Add-L].

- Given: the root of the derivation is [Add-L]:

$e1 \Rightarrow e1'$

 $e1 + e2 \Rightarrow e1' + e2$

- To prove: $\text{size } (e1' + e2) < \text{size } (e1 + e2)$
- IH: $\text{size } e1' < \text{size } e1$

```
size (e1' + e2)
= -- def. size
  size e1' + size e2
< -- IH
  size e1 + size e2
= -- def. size
  size (e1 + e2)
```

Inductive case [Add-R]. Try at home

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Theorem: Termination

Lemma 2: For any e, e' such that $e \Rightarrow e'$, $\text{size } e' < \text{size } e$.

Base case [Let].

- Given: the root of the derivation
is [Let]: $\text{let } x = v \text{ in } e2 \Rightarrow e2[x := v]$
- To prove: $\text{size } (e2[x := v]) < \text{size } (\text{let } x = v \text{ in } e2)$

```
size (e2[x := v])
= -- auxiliary lemma!
  size e2
< -- lemma
  size v + size e2
= -- def. size
  size (let x = v in e2)
```

Inductive case [Let-Def]. Try at home

QED.

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QUIZ

What is the IH for the inductive case [Let-Def]? *

$e1 \Rightarrow e1'$
[Let-Def] -----
 $\text{let } x = e1 \text{ in } e2 \Rightarrow \text{let } x = e1' \text{ in } e2$

- (A) $e1 \Rightarrow e1'$
- (B) $\text{size } e1' < \text{size } e1$
- (C) $\text{size } (\text{let } x = e1 \text{ in } e2) < \text{size } (\text{let } x = e1' \text{ in } e2)$



<http://tiny.cc/cse116-induct-ind>

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QUIZ

What is the IH for the inductive case [Let-Def]? *

$e1 \Rightarrow e1'$
[Let-Def] -----
 $\text{let } x = e1 \text{ in } e2 \Rightarrow \text{let } x = e1' \text{ in } e2$

- (A) $e1 \Rightarrow e1'$
- (B) $\text{size } e1' < \text{size } e1$
- (C) $\text{size } (\text{let } x = e1 \text{ in } e2) < \text{size } (\text{let } x = e1' \text{ in } e2)$



<http://tiny.cc/cse116-induct-grp>

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Nano2: adding functions

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Syntax

We need to extend the syntax of expressions and values:

```
e ::= n | x           -- expressions
   | e1 + e2
   | let x = e1 in e2
   | \x -> e         -- abstraction
   | e1 e2          -- application

v ::= n             -- values
   | \x -> e        -- abstraction
```

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Operational semantics

We need to extend our reduction relation with rules for abstraction and application:

```
          e1 => e1'
[App-L] -----
e1 e2 => e1' e2

          e => e'
[App-R] -----
v e => v e'

[App]  (\x -> e) v => e[x := v]
```

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QUIZ

With rules defined above, which reductions are valid? *

- (A) $(\lambda x y \rightarrow x + y) 1 (1 + 2) \Rightarrow (\lambda x y \rightarrow x + y) 1 3$
- (B) $(\lambda x y \rightarrow x + y) 1 (1 + 2) \Rightarrow (\lambda y \rightarrow 1 + y) (1 + 2)$
- (C) $(\lambda y \rightarrow 1 + y) (1 + 2) \Rightarrow (\lambda y \rightarrow 1 + y) 3$
- (D) $(\lambda y \rightarrow 1 + y) (1 + 2) \Rightarrow 1 + 1 + 2$
- (E) B and C



<http://tiny.cc/cse116-reduce2-ind>

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QUIZ

With rules defined above, which reductions are valid? *

- (A) $(\lambda x y \rightarrow x + y) 1 (1 + 2) \Rightarrow (\lambda x y \rightarrow x + y) 1 3$
- (B) $(\lambda x y \rightarrow x + y) 1 (1 + 2) \Rightarrow (\lambda y \rightarrow 1 + y) (1 + 2)$
- (C) $(\lambda y \rightarrow 1 + y) (1 + 2) \Rightarrow (\lambda y \rightarrow 1 + y) 3$
- (D) $(\lambda y \rightarrow 1 + y) (1 + 2) \Rightarrow 1 + 1 + 2$
- (E) B and C



<http://tiny.cc/cse116-reduce2-grp>

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Evaluation Order

```
((\lambda x y \rightarrow x + y) 1) (1 + 2)
=> (\lambda y \rightarrow 1 + y) (1 + 2)    -- [App-L], [App]
=> (\lambda y \rightarrow 1 + y) 3       -- [App-R], [Add]
=> 1 + 3                          -- [App]
=> 4                               -- [Add]
```

Our rules define **call-by-value**:

1. Evaluate the function (to a lambda)
2. Evaluate the argument (to some value)
3. "Make the call": make a substitution of formal to actual in the body of the lambda

The alternative is **call-by-name**:

- do not evaluate the argument before "making the call"
- can we modify the application rules for Nano2 to make it call-by-name?

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Theorems about Nano2

Let's prove something about Nano2!

1. Every Nano2 program terminates (?)
2. Closed Nano2 programs don't get stuck (?)

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QUIZ

Let's prove something about Nano2!

1. Every Nano2 program terminates (?)
2. Closed Nano2 programs don't get stuck (?)

Are these theorems still true? *

- (A) Both true
- (B) 1 is true, 2 is false
- (C) 1 is false, 2 is true
- (D) Both false



<http://tiny.cc/cse116-nano2-ind>

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QUIZ

Let's prove something about Nano2!

1. Every Nano2 program terminates (?)
2. Closed Nano2 programs don't get stuck (?)

Are these theorems still true? *

- (A) Both true
- (B) 1 is true, 2 is false
- (C) 1 is false, 2 is true
- (D) Both false



<http://tiny.cc/cse116-nano2-grp>

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Theorems about Nano2

1. Every Nano2 program terminates (?)

What about $(\lambda x \rightarrow x \ x) (\lambda x \rightarrow x \ x)$?

2. Closed Nano2 programs don't get stuck (?)

What about 1 2?

Both theorems are now false!

To recover these properties, we need to add *types*:

1. Every *well-typed* Nano2 program terminates
2. *Well-typed* Nano2 programs don't get stuck

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