

CSE 114A: Fall 2021

**Foundations of Programming
Languages**

Environments and closures

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Roadmap

Past weeks:

- How do we *use* a functional language?

Next weeks:

- How do we *implement* a functional language?
- ... in a functional language (of course)

WHY??

- ***Master*** the concepts of functional languages by implementing them!
- ***Practice*** problem solving using Haskell

This week: Interpreter

- How do we *evaluate* a program given its abstract syntax tree (AST)?
- How do we *prove properties* about our interpreter (e.g. that certain programs never crash)?

The Nano Language

Features of Nano:

1. Arithmetic expressions
2. Variables and let-bindings
3. Functions
4. Recursion

Reminder: Calculator

Arithmetic expressions:

$e ::= n$
| $e1 + e2$
| $e1 - e2$
| $e1 * e2$

Example:

$4 + 13$

$\Rightarrow 17$

Reminder: Calculator

Haskell datatype to *represent* arithmetic expressions:

```
data Expr = Num Int
          | Add Expr Expr
          | Sub Expr Expr
          | Mul Expr Expr
```

Haskell function to *evaluate* an expression:

```
eval :: Expr -> Int
eval (Num n)      = n
eval (Add e1 e2) = eval e1 + eval e2
eval (Sub e1 e2) = eval e1 - eval e2
eval (Mul e1 e2) = eval e1 * eval e2
```

Reminder: Calculator

Alternative representation:

```
data Binop = Add | Sub | Mul
```

```
data Expr = Num Int           -- number
           | Bin Binop Expr Expr -- binary expression
```

Evaluator for alternative representation:

```
eval :: Expr -> Int
eval (Num n)           = n
eval (Bin Add e1 e2)  = eval e1 + eval e2
eval (Bin Sub e1 e2)  = eval e1 - eval e2
eval (Bin Mul e1 e2)  = eval e1 * eval e2
```

The Nano Language

Features of Nano:

1. Arithmetic expressions [**done**]
2. Variables and let-bindings
3. Functions
4. Recursion

Extension: variables

Let's add variables and **let** bindings!

```
e ::= n | x
    | e1 + e2 | e1 - e2 | e1 * e2
    | let x = e1 in e2
```

Example:

```
let x = 4 + 13 in  -- 17
```

```
let y = 7 - 5 in  -- 2
```

```
x * y
```

```
==> 34
```


Extension: variables

Haskell representation:

```
data Expr = Num Int           -- number
          | ???               -- variable
          | Bin Binop Expr Expr -- binary expression
          | ???               -- let expression
```

Extension: variables

```
type Id = String
```

```
data Expr = Num Int           -- number
          | Var Id            -- variable
          | Bin Binop Expr Expr -- binary expression
          | Let Id Expr Expr  -- let expression
```

Haskell function to *evaluate* an expression:

```
eval :: Expr -> Int
```

```
eval (Num n)      = n
```

```
eval (Var x)      = ???
```

```
...
```

Extension: variables

```
type Id = String
```

```
data Expr = Num Int -- number
```

```
Var Id -- variable
```

How do we evaluate a variable?

We have to remember
which *value* it was bound to!

Haskell function

```
eval :: Expr
```

```
eval (Num n)
```

```
eval (Var x) = ???
```

```
...
```

Environment

An expression is evaluated in an **environment**, which maps all its *free variables* to *values*

Examples:

`x * y`

`= [x:17, y:2] => 34`

`x * y`

`= [x:17] => Error: unbound variable y`

`x * (let y = 2 in y)`

`= [x:17] => 34`

- How should we represent the environment?
- Which operations does it support?

Extension: variables

What does this evaluate to? *

```
let x = 5 in  
let y = x + z in  
let z = 10 in
```

y

- (A) 15
- (B) 5
- (C) Error: unbound variable x
- (D) Error: unbound variable y
- (E) Error: unbound variable z



<http://tiny.cc/cse116-vars-ind>

Extension: variables

What does this evaluate to? *

```
let x = 5 in  
let y = x + z in  
let z = 10 in
```

y

- (A) 15
- (B) 5
- (C) Error: unbound variable x
- (D) Error: unbound variable y
- (E) Error: unbound variable z



<http://tiny.cc/cse116-vars-grp>

Environment: API

To evaluate `let x = e1 in e2` in env:

- evaluate `e2` in an **extended** environment `env + [x:v]`
- where `v` is the result of evaluating `e1`

To evaluate `x` in env:

- **lookup** the most recently added binding for `x`

```
type Value = Int
```

```
data Env = ... -- representation not that important
```

```
-- | Add a new binding
```

```
add :: Id -> Value -> Env -> Env
```

```
-- | Lookup the most recently added binding
```

```
lookup :: Id -> Env -> Value
```

Evaluating expressions

Back to our expressions... now with environments!

```
data Expr = Num Int           -- number
          | Var Id            -- variable
          | Bin Binop Expr Expr -- binary expression
          | Let Id Expr Expr  -- let expression
```


Evaluating expressions

Haskell function to *evaluate* an expression:

```
eval :: Env -> Expr -> Value
eval env (Num n)           = n
eval env (Var x)           = lookup x env
eval env (Bin op e1 e2)   = f v1 v2
  where
    v1 = eval env e1
    v2 = eval env e2
    f = case op of
        Add -> (+)
        Sub -> (-)
        Mul -> (*)
eval env (Let x e1 e2)    = eval env' e2
  where
    v      = eval env e1
    env'  = add x v env
```

Example evaluation

Nano expression

```
let x = 1 in
let y = (let x = 2 in x) + x in
let x = 3 in
x + y
```

is represented in Haskell as:

```
exp1 = Let "x"
      (Num 1)
      (Let "y"
        (Add
          (Let "x" (Num 2) (Var x))
          (Var x))
        (Let "x"
          (Num 3)
          (Add (Var x) (Var y))))))
```

The diagram illustrates the nested structure of the Haskell code. It shows the following components:

- exp2**: A box containing the innermost expression `(Num 1)`.
- exp3**: A box containing the expression `(Add (Let "x" (Num 2) (Var x)) (Var x))`.
- exp4**: A box containing the expression `(Let "x" (Num 3) (Add (Var x) (Var y)))`.
- exp5**: A box containing the expression `(Let "x" (Num 3) (Add (Var x) (Var y)))`.

The overall structure is a nested list of expressions, with the outermost expression being `Let "x" (Num 1) (Let "y" (Add (Let "x" (Num 2) (Var x)) (Var x)) (Let "x" (Num 3) (Add (Var x) (Var y)))))`.

Example evaluation

```
eval [] exp1
=> eval [] (Let "x" (Num 1) exp2)
=> eval [("x",eval [] (Num 1))] exp2
=> eval [("x",1)]
    (Let "y" (Add exp3 exp4) exp5)
=> eval [("y",(eval [("x",1)] (Add exp3 exp4))), ("x",1)]
    exp5
=> eval [("y",(eval [("x",1)] (Let "x" (Num 2) (Var "x")))
        + eval [("x",1)] (Var "x"))), ("x",1)]
    exp5
=> eval [("y",(eval [("x",2), ("x",1)] (Var "x") -- new binding for x
        + 1)), ("x",1)]
    exp5
=> eval [("y",(2 -- use latest binding for x
        + 1)), ("x",1)]
    exp5
=> eval [("y",3), ("x",1)]
    (Let "x" (Num 3) (Add (Var "x") (Var "y")))
```

Example evaluation

```
=> eval [("y",3), ("x",1)]
      (Let "x" (Num 3) (Add (Var "x") (Var "y")))
=> eval [("x",3), ("y",3), ("x",1)]          -- new binding for x
      (Add (Var "x") (Var "y"))
=> eval [("x",3), ("y",3), ("x",1)] (Var "x")
+ eval [("x",3), ("y",3), ("x",1)] (Var "y")
=> 3 + 3
=> 6
```

Example evaluation

Same evaluation in a simplified format (Haskell `Expr` terms replaced by their “pretty-printed version”):

```
eval []
  {let x = 1 in let y = (let x = 2 in x) + x in let x = 3 in x + y}
=> eval [x:(eval [] 1)]
      {let y = (let x = 2 in x) + x in let x = 3 in x + y}
=> eval [x:1]
      {let y = (let x = 2 in x) + x in let x = 3 in x + y}
=> eval [y:(eval [x:1] {(let x = 2 in x) + x}), x:1]
      {let x = 3 in x + y}
=> eval [y:(eval [x:1] {let x = 2 in x} + eval [x:1] {x}), x:1]
      {let x = 3 in x + y}
      -- new binding for x:
=> eval [y:(eval [x:2,x:1] {x} + eval [x:1] {x}), x:1]
      {let x = 3 in x + y}
      -- use latest binding for x:
=> eval [y:(          2 + eval [x:1] {x}), x:1]
      {let x = 3 in x + y}
=> eval [y:(          2 + 1) , x:1]
      {let x = 3 in x + y}
```

Example evaluation

```
=> eval [y:(                2                + 1)                , x:1]
{let x = 3 in x + y}
=> eval [y:3, x:1]
{let x = 3 in x + y}
  -- new binding for x:
=> eval [x:3, y:3, x:1]
{x + y}
=> eval [x:3, y:3, x:1] x + eval [x:3, y:3, x:1] y
  -- use latest binding for x:
=> 3 + 3
=> 6
```

Runtime errors

Haskell function to *evaluate* an expression:

```
eval :: Env -> Expr -> Value
eval env (Num n)           = n
eval env (Var x)           = lookup x env -- can fail!
eval env (Bin op e1 e2)    = f v1 v2
  where
    v1 = eval env e1
    v2 = eval env e2
    f = case op of
         Add -> (+)
         Sub -> (-)
         Mul -> (*)
eval env (Let x e1 e2)     = eval env' e2
  where
    v      = eval env e1
    env'   = add x v env
```

How do we make sure lookup doesn't cause a run-time error?

Free vs bound variables

In `eval env e`, `env` must contain bindings for *all free variables* of `e`!

- an occurrence of `x` is **free** if it is not **bound**
- an occurrence of `x` is **bound** if it's inside `e2` where **let** `x = e1` **in** `e2`
- evaluation succeeds when an expression is **closed**!

QUIZ

Which variables are free in the expression? *

```
let y = (let x = 2 in x) + x in
```

```
let x = 3 in
```

```
x + y
```

- (A) None
- (B) x
- (C) y
- (D) x y



<http://tiny.cc/cse116-free-ind>

QUIZ

Which variables are free in the expression? *

```
let y = (let x = 2 in x) + x in
```

```
let x = 3 in
```

```
x + y
```

- (A) None
- (B) x
- (C) y
- (D) x y



<http://tiny.cc/cse116-free-grp>

The Nano Language

Features of Nano:

1. Arithmetic expressions [**done**]
2. Variables and let-bindings [**done**]
3. Functions
4. Recursion

Extension: functions

Let's add lambda abstraction and function application!

```
e ::= n | x
    | e1 + e2 | e1 - e2 | e1 * e2
    | let x = e1 in e2
    | \x -> e    -- abstraction
    | e1 e2      -- application
```

Example:

```
let c = 42 in
let cTimes = \x -> c * x in
cTimes 2
==> 84
```

Extension: functions

Haskell representation:

```
data Expr = Num Int           -- number
          | Var Id            -- variable
          | Bin Binop Expr Expr -- binary expression
          | Let Id Expr Expr  -- let expression
          | ???               -- abstraction
          | ???               -- application
```

Extension: functions

Haskell representation:

```
data Expr = Num Int           -- number
          | Var Id             -- variable
          | Bin Binop Expr Expr -- binary expression
          | Let Id Expr Expr   -- let expression
          | Lam Id Expr        -- abstraction
          | App Expr Expr      -- application
```

Extension: functions

Example:

```
let c = 42 in
let cTimes = \x -> c * x in
cTimes 2
```

represented as:

```
Let "c"
  (Num 42)
  (Let "cTimes"
    (Lam "x" (Mul (Var "c") (Var "x"))))
    (App (Var "cTimes") (Num 2)))
```

Extension: functions

Example:

```
let c = 42 in
let cTimes = \x -> c * x in
cTimes 2
```

How should we evaluate this expression?

```
eval []
  {let c = 42 in let cTimes = \x -> c * x in cTimes 2}
=> eval [c:42]
      {let cTimes = \x -> c * x in cTimes 2}
=> eval [cTimes:???, c:42]
                                             {cTimes 2}
```

What is the value of cTimes???

Rethinking our values

Until now: a program *evaluates* to an integer (or fails)

```
type Value = Int
```

```
type Env = [(Id, Value)]
```

```
eval :: Env -> Expr -> Value
```

Rethinking our values

What do these programs evaluate to?

(1)

```
\x -> 2 * x
```

```
==> ???
```

(2)

```
let f = \x -> \y -> 2 * (x + y) in
```

```
f 5
```

```
==> ???
```

Conceptually, (1) evaluates to itself (not exactly, see later). while (2) evaluates to something equivalent to $\lambda y. 2 * (5 + y)$

Rethinking our values

Now: a program evaluates to an integer or a *lambda abstraction* (or fails)

- Remember: functions are *first-class* values

Let's change our definition of values!

```
data Value = VNum Int
           | VLam ??? -- What info do we need to store?
```

```
-- Other types stay the same
```

```
type Env = [(Id, Value)]
```

```
eval :: Env -> Expr -> Value
```

Function values

How should we represent a function value?

```
let c = 42 in
```

```
let cTimes = \x -> c * x in
```

```
cTimes 2
```

We need to store enough information about `cTimes` so that we can later evaluate any *application* of `cTimes` (like `cTimes 2`)!

First attempt:

```
data Value = VNum Int
           | VLam Id Expr -- formal + body
```

Function values

Let's try this!

```
eval []
  {let c = 42 in let cTimes = \x -> c * x in cTimes 2}
=> eval [c:42]
      {let cTimes = \x -> c * x in cTimes 2}
=> eval [cTimes:(\x -> c*x), c:42]
      {cTimes 2}
  -- evaluate the function:
=> eval [cTimes:(\x -> c*x), c:42]
      {(\x -> c * x) 2}
  -- evaluate the argument, bind to x, evaluate body:
=> eval [x:2, cTimes:(\x -> c*x), c:42]
      {c * x}
=>
      42 * 2
=>
      84
```

Looks good... can you spot a problem?

QUIZ

What should this evaluate to? *

```
let c = 42 in
```

```
let cTimes = \x -> c * x in -- but which c???
```

```
let c = 5 in
```

```
cTimes 2
```

- (A) 84
- (B) 10
- (C) Error: multiple definitions of c



<http://tiny.cc/cse116-cscope-ind>

QUIZ

What should this evaluate to? *

```
let c = 42 in
```

```
let cTimes = \x -> c * x in -- but which c???
```

```
let c = 5 in
```

```
cTimes 2
```

- (A) 84
- (B) 10
- (C) Error: multiple definitions of c



<http://tiny.cc/cse116-cscope-grp>

Static vs Dynamic Scoping

What we want:

```
let c = 42 in
```

```
let cTimes = \x -> c * x in
```

```
let c = 5 in
```

```
cTimes 2
```

```
=> 84
```

Lexical (or static) scoping:

- each occurrence of a variable refers to the most recent binding *in the program text*
- definition of each variable is unique and known *statically*
- good for readability and debugging: don't have to figure out where a variable got "assigned"

Static vs Dynamic Scoping

What we **don't** want:

```
let c = 42 in
let cTimes = \x -> c * x in
let c = 5 in
cTimes 2
=> 10
```

Dynamic scoping:

- each occurrence of a variable refers to the most recent binding *during program execution*
- can't tell where a variable is defined just by looking at the function body
- nightmare for readability and debugging:

Static vs Dynamic Scoping

Dynamic scoping:

- each occurrence of a variable refers to the most recent binding *during program execution*
- can't tell where a variable is defined just by looking at the function body
- nightmare for readability and debugging:

```
let cTimes = \x -> c * x in
let c = 5 in
let res1 = cTimes 2 in -- ==> 10
let c = 10 in
let res2 = cTimes 2 in -- ==> 20!!!
res2 - res1
```

Function values

```
data Value = VNum Int
           | VLam Id Expr -- formal + body
```

This representation can only implement dynamic scoping!

```
let c = 42 in
```

```
let cTimes = \x -> c * x in
```

```
let c = 5 in
```

```
cTimes 2
```

evaluates as:

```
eval []
```

```
{let c = 42 in let cTimes = \x -> c * x in let c = 5 in cTimes 2}
```

Function values

```
eval []
{let c = 42 in let cTimes = \x -> c * x in let c = 5 in cTimes 2}
=> eval [c:42]
      {let cTimes = \x -> c * x in let c = 5 in cTimes 2}
=> eval [cTimes:(\x -> c*x), c:42]
      {let c = 5 in cTimes 2}
=> eval [c:5, cTimes:(\x -> c*x), c:42]
      {cTimes 2}
=> eval [c:5, cTimes:(\x -> c*x), c:42]
      {(\x -> c * x) 2}
=> eval [x:2, c:5, cTimes:(\x -> c*x), c:42]
      {c * x}
      -- latest binding for c is 5!
=>
      5 * 2
=>
      10
```

Lesson learned: need to remember what `c` was bound to when `cTimes` was defined!

- i.e. “freeze” the environment at function definition

Closures

To implement lexical scoping, we will represent function values as *closures*

Closure = *lambda abstraction* (formal + body) + *environment* at function definition

```
data Value = VNum Int
           | VClos Env Id Expr -- env + formal + body
```

Closures

Our example:

```
eval []
{let c = 42 in let cTimes = \x -> c * x in let c = 5 in cTimes 2}
=> eval [c:42]
      {let cTimes = \x -> c * x in let c = 5 in cTimes 2}
  -- remember current env:
=> eval [cTimes:<[c:42], \x -> c*x>, c:42]
      {let c = 5 in cTimes 2}
=> eval [c:5, cTimes:<[c:42], \x -> c*x>, c:42]
      {cTimes 2}
=> eval [c:5, cTimes:<[c:42], \x -> c*x>, c:42]
      {<[c:42], \x -> c * x> 2}
  -- restore env to the one inside the closure, then bind 2 to x:
=> eval [x:2, c:42]
      {c * x}
=>
      42 * 2
=>
      84
```

QUIZ

Which variables should be saved in the closure environment of f? *

```
let a = 20 in
let f =
  \x -> let y = x + 1 in
        let g = \z -> y + z in
        a + g x
in ...
```

- (A) a
- (B) a x
- (C) y g
- (D) a y g
- (E) a x y g z



<http://tiny.cc/cse116-env-ind>

QUIZ

Which variables should be saved in the closure environment of f? *

```
let a = 20 in
let f =
  \x -> let y = x + 1 in
        let g = \z -> y + z in
        a + g x
in ...
```

- (A) a
- (B) a x
- (C) y g
- (D) a y g
- (E) a x y g z



<http://tiny.cc/cse116-env-grp>

Free vs bound variables

- An occurrence of x is **free** if it is not **bound**
- An occurrence of x is **bound** if it's inside
 - e_2 where **let** $x = e_1$ **in** e_2
 - e where $\backslash x \rightarrow e$
- A closure environment has to save *all free variables* of a function definition!

```
let a = 20 in
```

```
let f =
```

```
  \x -> let y = x + 1 in
```

```
    let g = \z -> y + z in
```

```
    a + g x -- a is the only free variable!
```

```
in ...
```

Evaluator

Let's modify our evaluator to handle functions!

```
data Value = VNum Int
           | VClos Env Id Expr -- env + formal + body

eval :: Env -> Expr -> Value
eval env (Num n)          = VNum n -- must wrap in VNum now!
eval env (Var x)          = lookup x env
eval env (Bin op e1 e2) = VNum (f v1 v2)
  where
    (VNum v1) = eval env e1
    (VNum v2) = eval env e2
    f = ... -- as before
eval env (Let x e1 e2) = eval env' e2
  where
    v = eval env e1
    env' = add x v env
eval env (Lam x body) = ??? -- construct a closure
eval env (App fun arg) = ??? -- eval fun, then arg, then apply
```

Evaluator

Evaluating functions:

- **Construct a closure:** save environment at function definition
- **Apply a closure:** restore saved environment, add formal, evaluate the body

```
eval :: Env -> Expr -> Value
```

```
...
```

```
eval env (Lam x body) = VClos env x body
```

```
eval env (App fun arg) = eval bodyEnv body
```

```
  where
```

```
    (VClos closEnv x body) = eval env fun -- eval function to closure
```

```
    vArg                    = eval env arg -- eval argument
```

```
    bodyEnv                 = add x vArg closEnv
```

Evaluator

Evaluating functions:

- **Construct a closure:** save environment at function definition
- **Apply a closure:** restore saved environment, add formal, evaluate the body

```
eval :: Env -> Expr -> Value
```

```
...
```

```
eval env (Lam x body) = VClos env x body
```

```
eval env (App fun arg) =
```

```
  let vArg = eval env arg in -- eval argument
```

```
  let (VClos closEnv x body) = (eval env fun) in
```

```
  let bodyEnv = add x vArg closEnv in
```

```
  eval bodyEnv body
```

Quiz

With eval as defined above, what does this evaluate to? *

```
let f = \x -> x + y in
let y = 10 in
f 5
```

- (A) 15
- (B) 5
- (C) Error: unbound variable x
- (D) Error: unbound variable y
- (E) Error: unbound variable f



<http://tiny.cc/cse116-enveval-ind>

Quiz

With eval as defined above, what does this evaluate to? *

```
let f = \x -> x + y in
let y = 10 in
f 5
```

- (A) 15
- (B) 5
- (C) Error: unbound variable x
- (D) Error: unbound variable y
- (E) Error: unbound variable f



<http://tiny.cc/cse116-enveval-grp>

Evaluator

```
eval []
```

```
  {let f = \x -> x + y in let y = 10 in f 5}
```

```
=> eval [f:<[], \x -> x + y>]
```

```
      {let y = 10 in f 5}
```

```
=> eval [y:10, f:<[], \x -> x + y>]
```

```
      {f 5}
```

```
=> eval [y:10, f:<[], \x -> x + y>]
```

```
      {<[], \x -> x + y> 5}
```

```
=> eval [x:5] -- env got replaced by closure env + formal!
```

```
      {x + y} -- y is unbound!
```

Quiz

With eval as defined above, what does this evaluate to? *

```
let f = \n -> n * f (n - 1) in  
f 5
```

- (A) 120
- (B) Evaluation does not terminate
- (C) Error: unbound variable f



<http://tiny.cc/cse116-enveval2-ind>

Quiz

With eval as defined above, what does this evaluate to? *

```
let f = \n -> n * f (n - 1) in  
f 5
```

- (A) 120
- (B) Evaluation does not terminate
- (C) Error: unbound variable f



<http://tiny.cc/cse116-enveval2-grp>

Evaluator

```
eval []
      {let f = \n -> n * f (n - 1) in f 5}
=> eval [f:<[], \n -> n * f (n - 1)>]
                                     {f 5}
=> eval [f:<[], \n -> n * f (n - 1)>]
      {<[], \n -> n * f (n - 1)> 5}
=> eval [n:5] -- env got replaced by closure env + formal!
              {n * f (n - 1)} -- f is unbound!
```

Lesson learned: to support recursion, we need a different way of constructing the closure environment!