CSE 114A: Fall 2021

Foundations of Programming Languages

Environments and closures

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Roadmap

Past weeks:

- How do we *use* a functional language?

Next weeks:

- How do we *implement* a functional language?
- ... in a functional language (of course)

WHY??

- *Master* the concepts of functional languages by implementing them!
- *Practice* problem solving using Haskell

This week: Interpreter

- How do we *evaluate* a program given its abstract syntax tree (AST)?
- How do we *prove properties* about our interpreter (e.g. that certain programs never crash)?

The Nano Language

Features of Nano:

1. Arithmetic expressions
2. Variables and let-bindings
3. Functions
4. Recursion

Reminder: Calculator

Arithmetic expressions:

```
e ::= n
    | e1 + e2
    | e1 - e2
    | e1 * e2
```

Example:

```
4 + 13
==> 17
```

Reminder: Calculator

Haskell datatype to *represent* arithmetic expressions:

```
data Expr = Num Int
          | Add Expr Expr
          | Sub Expr Expr
          | Mul Expr Expr
```

Haskell function to *evaluate* an expression:

```
eval :: Expr -> Int
eval (Num n)     = n
eval (Add e1 e2) = eval e1 + eval e2
eval (Sub e1 e2) = eval e1 - eval e2
eval (Mul e1 e2) = eval e1 * eval e2
```

Reminder: Calculator

Alternative representation:

```
data Binop = Add | Sub | Mul

data Expr = Num Int              -- number
          | Bin Binop Expr Expr  -- binary expression
```

Evaluator for alternative representation:

```
eval :: Expr -> Int
eval (Num n)         = n
eval (Bin Add e1 e2) = eval e1 + eval e2
eval (Bin Sub e1 e2) = eval e1 - eval e2
eval (Bin Mul e1 e2) = eval e1 * eval e2
```

The Nano Language

Features of Nano:

1. Arithmetic expressions **[done]**
2. Variables and let-bindings
3. Functions
4. Recursion

Extension: variables

Let's add variables and **let** bindings!

```
e ::= n | x
    | e1 + e2 | e1 - e2 | e1 * e2
    | let x = e1 in e2
```

Example:

```
let x = 4 + 13 in   -- 17
let y = 7 - 5 in    -- 2
x * y

==> 34
```

Extension: variables

Haskell representation:

```
data Expr = Num Int            -- number
          | ???                -- variable
          | Bin Binop Expr Expr -- binary expression
          | ???                -- Let expression
```

Extension: variables

```haskell
type Id = String

data Expr = Num Int          -- number
          | Var Id           -- variable
          | Bin Binop Expr Expr  -- binary expression
          | Let Id Expr Expr     -- let expression
```

Haskell function to *evaluate* an expression:

```haskell
eval :: Expr -> Int
eval (Num n)        = n
eval (Var x)        = ???
...
```

Extension: variables

How do we evaluate a variable?

We have to remember which *value* it was bound to!

Environment

An expression is evaluated in an **environment**, which maps all its *free variables* to *values*

Examples:

```
x * y
=[x:17, y:2]=> 34
```

- How should we represent the environment?
- Which operations does it support?

```
x * y
=[x:17]=> Error: unbound variable y


x * (let y = 2 in y)
=[x:17]=> 34
```

Extension: variables

What does this evaluate to? *

```
let x = 5 in
let y = x + z in
let z = 10 in
y
```

○ (A) 15

○ (B) 5

○ (C) Error: unbound variable x

○ (D) Error: unbound variable y

○ (E) Error: unbound variable z

http://tiny.cc/cse116-vars-ind

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Extension: variables

What does this evaluate to? *

```
let x = 5 in
let y = x + z in
let z = 10 in
y
```

○ (A) 15

○ (B) 5

○ (C) Error: unbound variable x

○ (D) Error: unbound variable y

○ (E) Error: unbound variable z

http://tiny.cc/cse116-vars-grp

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Environment: API

To evaluate `let x = e1 in e2` in env:
- evaluate e2 in an **extended** environment `env + [x:v]`
- where `v` is the result of evaluating `e1`

To evaluate `x` in env:
- **lookup** the most recently added binding for `x`

```
type Value = Int

data Env = ... -- representation not that important

-- | Add a new binding
add    :: Id -> Value -> Env -> Env

-- | Lookup the most recently added binding
lookup :: Id -> Env -> Value
```

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Evaluating expressions

Back to our expressions... now with environments!

```haskell
data Expr = Num Int           -- number
          | Var Id            -- variable
          | Bin Binop Expr Expr  -- binary expression
          | Let Id Expr Expr  -- let expression
```

Evaluating expressions

Haskell function to *evaluate* an expression:

```haskell
eval :: Env -> Expr -> Value
eval env (Num n)        = n
eval env (Var x)        = lookup x env
eval env (Bin op e1 e2) = f v1 v2
  where
    v1 = eval env e1
    v2 = eval env e2
    f = case op of
          Add -> (+)
          Sub -> (-)
          Mul -> (*)
eval env (Let x e1 e2)  = eval env' e2
  where
    v    = eval env e1
    env' = add x v env
```

Example evaluation

Nano expression

```haskell
let x = 1 in
let y = (let x = 2 in x) + x in
let x = 3 in
x + y
```

is represented in Haskell as:

```haskell
exp1 =   Let "x"
             (Num 1)                      exp2
             (Let "y"
                 (Add                     exp3
                     (Let "x" (Num 2) (Var x))
                  exp4 (Var x))
                 (Let "x"                 exp5
                     (Num 3)
                     (Add (Var x) (Var y))))
```

Example evaluation

```
eval [] exp1
=> eval []                          (Let "x" (Num 1) exp2)
=> eval [("x",eval [] (Num 1))] exp2
=> eval [("x",1)]
   (Let "y" (Add exp3 exp4) exp5)
=> eval [("y",(eval [("x",1)] (Add exp3 exp4))), ("x",1)]
    exp5
=> eval [("y",(eval [("x",1)] (Let "x" (Num 2) (Var "x"))
          + eval [("x",1)] (Var "x"))), ("x",1)]
    exp5
=> eval [("y",(eval [("x",2), ("x",1)] (Var "x") -- new binding for x
          + 1)), ("x",1)]
    exp5
=> eval [("y",(2  -- use latest binding for x
          + 1)), ("x",1)]
    exp5
=> eval [("y",3), ("x",1)]
   (Let "x" (Num 3) (Add (Var "x") (Var "y")))
```

Example evaluation

```
=> eval [("y",3), ("x",1)]
   (Let "x" (Num 3) (Add (Var "x") (Var "y")))
=> eval [("x",3), ("y",3), ("x",1)]         -- new binding for x
   (Add (Var "x") (Var "y"))
=> eval [("x",3), ("y",3), ("x",1)] (Var "x")
 + eval [("x",3), ("y",3), ("x",1)] (Var "y")
=> 3 + 3
=> 6
```

Example evaluation

Same evaluation in a simplified format (Haskell `Expr` terms replaced by their "pretty-printed version"):

```
   eval []
   {let x = 1 in let y = (let x = 2 in x) + x in let x = 3 in x + y}
=> eval [x:(eval [] 1)]
                {let y = (let x = 2 in x) + x in let x = 3 in x + y}
=> eval [x:1]
                {let y = (let x = 2 in x) + x in let x = 3 in x + y}
=> eval [y:(eval [x:1] {(let x = 2 in x) + x}), x:1]
                                            {let x = 3 in x + y}
=> eval [y:(eval [x:1]  {let x = 2 in x} + eval [x:1] {x}), x:1]
                                            {let x = 3 in x + y}
        -- new binding for x:
=> eval [y:(eval [x:2,x:1] {x}       + eval [x:1] {x}), x:1]
                                            {let x = 3 in x + y}
        -- use latest binding for x:
=> eval [y:(              2           + eval [x:1] {x}), x:1]
                                            {let x = 3 in x + y}
=> eval [y:(              2           + 1)            , x:1]
                                            {let x = 3 in x + y}
```

Example evaluation

```
=> eval [y:(                    2              + 1)              , x:1]
                                     {let x = 3 in x + y}
=> eval [y:3, x:1]
                                     {let x = 3 in x + y}
   -- new binding for x:
=> eval [x:3, y:3, x:1]
                                              {x + y}
=> eval [x:3, y:3, x:1] x + eval [x:3, y:3, x:1] y
   -- use latest binding for x:
=> 3 + 3
=> 6
```

Runtime errors

Haskell function to *evaluate* an expression:

```haskell
eval :: Env -> Expr -> Value
eval env (Num n)      = n
eval env (Var x)      = lookup x env -- can fail!
eval env (Bin op e1 e2) = f v1 v2
  where
    v1 = eval env e1
    v2 = eval env e2
    f = case op of
          Add -> (+)
          Sub -> (-)
          Mul -> (*)
eval env (Let x e1 e2) = eval env' e2
  where
    v    = eval env e1
    env' = add x v env
```

How do we make sure `lookup` doesn't cause a run-time error?

Free vs bound variables

In `eval env e`, env must contain bindings for *all free variables* of e!

- an occurrence of x is **free** if it is not **bound**
- an occurrence of x is **bound** if it's inside e2 where `let x = e1 in e2`
- evaluation succeeds when an expression is **closed**!

QUIZ

Which variables are free in the expression? *

```
let y = (let x = 2 in x) + x in
let x = 3 in
x + y
```

○ (A) None

○ (B) x

○ (C) y

○ (D) x y

http://tiny.cc/cse116-free-ind

QUIZ

Which variables are free in the expression? *

```
let y = (let x = 2 in x) + x in
let x = 3 in
x + y
```

○ (A) None

○ (B) x

○ (C) y

○ (D) x y

http://tiny.cc/cse116-free-grp

The Nano Language

Features of Nano:

1. Arithmetic expressions **[done]**
2. Variables and let-bindings **[done]**
3. Functions
4. Recursion

Extension: functions

Let's add lambda abstraction and function application!

```
e ::= n | x
    | e1 + e2 | e1 - e2 | e1 * e2
    | let x = e1 in e2
    | \x -> e   -- abstraction
    | e1 e2     -- application
```

Example:
```
let c = 42 in
let cTimes = \x -> c * x in
cTimes 2
==> 84
```

Extension: functions

Haskell representation:
```
data Expr = Num Int         -- number
          | Var Id          -- variable
          | Bin Binop Expr Expr  -- binary expression
          | Let Id Expr Expr     -- let expression
          | ???             -- abstraction
          | ???             -- application
```

Extension: functions

Haskell representation:
```
data Expr = Num Int         -- number
          | Var Id          -- variable
          | Bin Binop Expr Expr  -- binary expression
          | Let Id Expr Expr     -- let expression
          | Lam Id Expr     -- abstraction
          | App Expr Expr   -- application
```

Extension: functions

Example:

```
let c = 42 in
let cTimes = \x -> c * x in
cTimes 2
```

represented as:

```
Let "c"
  (Num 42)
  (Let "cTimes"
    (Lam "x" (Mul (Var "c") (Var "x")))
    (App (Var "cTimes") (Num 2)))
```

Extension: functions

Example:

```
let c = 42 in
let cTimes = \x -> c * x in
cTimes 2
```

How should we evaluate this expression?

```
   eval []
     {let c = 42 in let cTimes = \x -> c * x in cTimes 2}
=> eval [c:42]
                 {let cTimes = \x -> c * x in cTimes 2}
=> eval [cTimes:???, c:42]
                                       {cTimes 2}
```

What is the **value** of cTimes???

Rethinking our values

Until now: a program *evaluates* to an integer (or fails)

```
type Value = Int

type Env = [(Id, Value)]

eval :: Env -> Expr -> Value
```

Rethinking our values

What do these programs evaluate to?

```
(1)
\x -> 2 * x
==> ???


(2)
let f = \x -> \y -> 2 * (x + y) in
f 5
==> ???
```

Conceptually, (1) evaluates to itself (not exactly, see later). while (2) evaluates to something equivalent to \y -> 2 * (5 + y)

Rethinking our values

Now: a program evaluates to an integer or *a lambda abstraction* (or fails)

- Remember: functions are *first-class* values

Let's change our definition of values!

```
data Value = VNum Int
           | VLam ??? -- What info do we need to store?

-- Other types stay the same
type Env = [(Id, Value)]

eval :: Env -> Expr -> Value
```

Function values

How should we represent a function value?

```
let c = 42 in
let cTimes = \x -> c * x in
cTimes 2
```

We need to store enough information about cTimes so that we can later evaluate any *application* of cTimes (like cTimes 2)!

First attempt:

```
data Value = VNum Int
           | VLam Id Expr -- formal + body
```

Function values

Let's try this!

```
eval []
  {let c = 42 in let cTimes = \x -> c * x in cTimes 2}
=> eval [c:42]
                {let cTimes = \x -> c * x in cTimes 2}
=> eval [cTimes:(\x -> c*x), c:42]
                                {cTimes 2}
  -- evaluate the function:
=> eval [cTimes:(\x -> c*x), c:42]
                                {(\x -> c * x) 2}
  -- evaluate the argument, bind to x, evaluate body:
=> eval [x:2, cTimes:(\x -> c*x), c:42]
                                {c * x}
=>                              42 * 2
=>                              84
```

Looks good... can you spot a problem?

QUIZ

What should this evaluate to? *

```
let c = 42 in
let cTimes = \x -> c * x in -- but which c???
let c = 5 in
cTimes 2
```

○ (A) 84

○ (B) 10

○ (C) Error: multiple definitions of c

http://tiny.cc/cse116-cscope-ind

QUIZ

What should this evaluate to? *

```
let c = 42 in
let cTimes = \x -> c * x in -- but which c???
let c = 5 in
cTimes 2
```

○ (A) 84

○ (B) 10

○ (C) Error: multiple definitions of c

http://tiny.cc/cse116-cscope-grp

Static vs Dynamic Scoping

What we want:

```
let c = 42 in
let cTimes = \x -> c * x in
let c = 5 in
cTimes 2
=> 84
```

Lexical (or **static**) scoping:

- each occurrence of a variable refers to the most recent binding *in the program text*
- definition of each variable is unique and known *statically*
- good for readability and debugging: don't have to figure out where a variable got "assigned"

Static vs Dynamic Scoping

What we **don't** want:

```
let c = 42 in
let cTimes = \x -> c * x in
let c = 5 in
cTimes 2
=> 10
```

Dynamic scoping:

- each occurrence of a variable refers to the most recent binding *during program execution*
- can't tell where a variable is defined just by looking at the function body
- nightmare for readability and debugging:

Static vs Dynamic Scoping

Dynamic scoping:

- each occurrence of a variable refers to the most recent binding *during program execution*
- can't tell where a variable is defined just by looking at the function body
- nightmare for readability and debugging:

```
let cTimes = \x -> c * x in
let c = 5 in
let res1 = cTimes 2 in -- ==> 10
let c = 10 in
let res2 = cTimes 2 in -- ==> 20!!!
res2 - res1
```

Function values

```
data Value = VNum Int
           | VLam Id Expr -- formal + body
```

This representation can only implement dynamic scoping!

```
let c = 42 in
let cTimes = \x -> c * x in
let c = 5 in
cTimes 2
```

evaluates as:

```
eval []
{let c = 42 in let cTimes = \x -> c * x in let c = 5 in cTimes 2}
```

Function values

```
eval []
  {let c = 42 in let cTimes = \x -> c * x in let c = 5 in cTimes 2}
=> eval [c:42]
                {let cTimes = \x -> c * x in let c = 5 in cTimes 2}
=> eval [cTimes:(\x -> c*x), c:42]
                                   {let c = 5 in cTimes 2}
=> eval [c:5, cTimes:(\x -> c*x), c:42]
                                        {cTimes 2}
=> eval [c:5, cTimes:(\x -> c*x), c:42]
                                          {(\x -> c * x) 2}
=> eval [x:2, c:5, cTimes:(\x -> c*x), c:42]
                                            {c * x}
  -- latest binding for c is 5!
=>                                          5 * 2
=>                                          10
```

Lesson learned: need to remember what `c` was bound to when `cTimes` was defined!

- i.e. "freeze" the environment at function definition

Closures

To implement lexical scoping, we will represent function values as *closures*

Closure = *lambda abstraction* (formal + body) + *environment* at function definition

```
data Value = VNum Int
           | VClos Env Id Expr -- env + formal + body
```

Closures

Our example:

```
   eval []
      {let c = 42 in let cTimes = \x -> c * x in let c = 5 in cTimes 2}
=> eval [c:42]
                  {let cTimes = \x -> c * x in let c = 5 in cTimes 2}
   -- remember current env:
=> eval [cTimes:<[c:42], \x -> c*x>, c:42]
                                    {let c = 5 in cTimes 2}
=> eval [c:5, cTimes:<[c:42], \x -> c*x>, c:42]
                                              {cTimes 2}
=> eval [c:5, cTimes:<[c:42], \x -> c*x>, c:42]
                                          {<[c:42], \x -> c * x> 2}
   -- restore env to the one inside the closure, then bind 2 to x:
=> eval [x:2, c:42]
                                              {c * x}
=>                                            42 * 2
=>                                            84
```

QUIZ

Which variables should be saved in the closure environment
of f? *

```
let a = 20 in
let f =
  \x -> let y = x + 1 in
        let g = \z -> y + z in
        a + g x
  in ...
```

- (A) a
- (B) a x
- (C) y g
- (D) a y g
- (E) a x y g z

http://tiny.cc/cse116-env-ind

QUIZ

Which variables should be saved in the closure environment
of f? *

```
let a = 20 in
let f =
  \x -> let y = x + 1 in
        let g = \z -> y + z in
        a + g x
  in ...
```

- (A) a
- (B) a x
- (C) y g
- (D) a y g
- (E) a x y g z

http://tiny.cc/cse116-env-grp

Free vs bound variables

- An occurrence of `x` is **free** if it is not **bound**
- An occurrence of `x` is **bound** if it's inside
  - `e2` where `let x = e1 in e2`
  - `e` where `\x -> e`
- A closure environment has to save *all free variables* of a function definition!

```
let a = 20 in
let f =
  \x -> let y = x + 1 in
        let g = \z -> y + z in
        a + g x -- a is the only free variable!
  in ...
```

Evaluator

Let's modify our evaluator to handle functions!

```
data Value = VNum Int
           | VClos Env Id Expr -- env + formal + body

eval :: Env -> Expr -> Value
eval env (Num n)        = VNum n -- must wrap in VNum now!
eval env (Var x)        = lookup x env
eval env (Bin op e1 e2) = VNum (f v1 v2)
  where
    (VNum v1) = eval env e1
    (VNum v2) = eval env e2
    f = ... -- as before
eval env (Let x e1 e2) = eval env' e2
  where
    v = eval env e1
    env' = add x v env
eval env (Lam x body) = ??? -- construct a closure
eval env (App fun arg) = ??? -- eval fun, then arg, then apply
```

Evaluator

Evaluating functions:

- **Construct a closure**: save environment at function definition
- **Apply a closure**: restore saved environment, add formal, evaluate the body

```
eval :: Env -> Expr -> Value
...
eval env (Lam x body) = VClos env x body
eval env (App fun arg) = eval bodyEnv body
  where
    (VClos closEnv x body) = eval env fun -- eval function to closure
    vArg                   = eval env arg -- eval argument
    bodyEnv                = add x vArg closEnv
```

Evaluator

Evaluating functions:

- **Construct a closure**: save environment at function definition
- **Apply a closure**: restore saved environment, add formal, evaluate the body

```haskell
eval :: Env -> Expr -> Value
...
eval env (Lam x body) = VClos env x body
eval env (App fun arg) =
    let vArg = eval env arg in -- eval argument
    let (VClos closEnv x body) = (eval env fun) in
    let bodyEnv = add x vArg closEnv in
    eval bodyEnv body
```

Quiz

With eval as defined above, what does this evaluate to? *

```
let f = \x -> x + y in
let y = 10 in
f 5
```

○ (A) 15

○ (B) 5

○ (C) Error: unbound variable x

○ (D) Error: unbound variable y

○ (E) Error: unbound variable f

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Quiz

With eval as defined above, what does this evaluate to? *

```
let f = \x -> x + y in
let y = 10 in
f 5
```

○ (A) 15

○ (B) 5

○ (C) Error: unbound variable x

○ (D) Error: unbound variable y

○ (E) Error: unbound variable f

http://tiny.cc/cse116-enveval-grp

Evaluator

```
  eval []
    {let f = \x -> x + y in let y = 10 in f 5}
=> eval [f:<[], \x -> x + y>]
                              {let y = 10 in f 5}
=> eval [y:10, f:<[], \x -> x + y>]
                                      {f 5}
=> eval [y:10, f:<[], \x -> x + y>]
                      {<[], \x -> x + y> 5}
=> eval [x:5] -- env got replaced by closure env + formal!
                      {x + y}  -- y is unbound!
```

Quiz

With eval as defined above, what does this evaluate to? *

```
let f = \n -> n * f (n - 1) in
f 5
```

○ (A) 120

○ (B) Evaluation does not terminate

○ (C) Error: unbound variable f

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Quiz

With eval as defined above, what does this evaluate to? *

```
let f = \n -> n * f (n - 1) in
f 5
```

○ (A) 120

○ (B) Evaluation does not terminate

○ (C) Error: unbound variable f

http://tiny.cc/cse116-enveval2-grp

Evaluator

```
  eval []
      {let f = \n -> n * f (n - 1) in f 5}
=> eval [f:<[], \n -> n * f (n - 1)>]
                                    {f 5}
=> eval [f:<[], \n -> n * f (n - 1)>]
          {<[], \n -> n * f (n - 1)> 5}
=> eval [n:5] -- env got replaced by closure env + formal!
                    {n * f (n - 1)} -- f is unbound!
```

Lesson learned: to support recursion, we need a different way of constructing the closure environment!